6.1 Completing the Square Discovery

MPM2D
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1 Completing the Square

1.1 Purpose

Completing the square is used to rewrite a quadratic function from standard form ($ax^2 + bx + c$) to vertex form ($a(x-h)^2 + k$). One may need to do this for several reasons:

1. When studying quadratic functions, the vertex form is useful because it gives the coordinates of the vertex with no work. The vertex of $f(x) = a(x-h)^2 + k$ is the point (h, k).
2. This is one of the ways to solve quadratic equations. It is also one of the ways to prove the quadratic formula.
3. In many cases (such as integration), it is desirable to be able to write a quadratic expression as a difference or a sum of two squares. This is what the vertex form is.

1.2 How to do it?

First, we look at the format of a perfect square.

$$\begin{aligned}(x+a)^2 &= x^2 + 2ax + a^2 \\ (x-a)^2 &= x^2 - 2ax + a^2\end{aligned}$$

When we complete the square, we are given a term in x^2 and a term in x. We are trying to find what is missing in order to have a perfect square. From the expression of a perfect square, we see that what we are missing is the square of half the coefficient of x. Indeed, the coefficient of x is $2a$, half of it is a. Its square is a^2. The perfect square will be $(x \pm a)^2$. The sign will depend on the sign of the coefficient of x.

Example 1 *What is x^2+4x missing to be a perfect square?*
Half the coefficient of x is 2. Its square is 4. So, x^2+4x is missing 4 to be a perfect square. We can verify this easily:

$$x^2+4x+4=(x+2)^2$$

Example 2 *What is x^2-8x missing to be a perfect square?*
Half the coefficient of x is 4. Its square is 16. So, x^2+4x is missing 16 to be a perfect square. We can verify this easily:

$$x^2-8x+16=(x-4)^2$$

Completing the square is simply finding the missing term in order to have a perfect square.

Given a quadratic function, to put it in vertex form, follow the following steps:

1. Factor the coefficient of x^2 from the terms in $x^2$and x. Don't do anything with the constant.

2. Complete the square on the terms in x^2 and x as shown above. Since adding the missing term would change the given expression, we add it and subtract it at the same time.

3. The first three terms will be a perfect square.

4. Distribute the coefficient factored in step 1.

1.3 Examples

Example 3 *Write $y=2x^2+12x+1$ in vertex form.*

$$\begin{aligned} y &= 2x^2+12x+1 \\ &= 2\left(x^2+6x\right)+1 \text{ we factored the coefficient of } x^2 \\ &= 2\left(x^2+6x+9-9\right)+1 \text{ we added and subtracted } \left(\frac{6}{2}\right)^2 \\ &= 2\left((x+3)^2-9\right)+1 \\ &= 2(x+3)^2-18+1 \\ &= 2(x+3)^2-17 \end{aligned}$$

Example 4 *Write $y=x^2-2x+3$ in vertex form.*

$$\begin{aligned} y &= x^2-2x+3 \\ &= \left(x^2-2x+1-1\right)+3 \\ &= \left(x^2-2x+1\right)-1+3 \\ &= (x-1)^2+2 \end{aligned}$$

1.4 Problems

1. Write each equation below in vertex form

   (a) $y = 2x^2 + 12x - 2$ (answer: $y = 2(x+3)^2 - 20$)

   (b) $y = x^2 - 2x$ (answer: $y = (x-1)^2 - 1$)

   (c) $y = -2x^2 - 12x - 23$ (answer: $y = -2(x+3)^2 - 5$)

a) $y = 2x^2 + 12x - 2$

$= (2x^2 + 12x) - 2$

$= 2(x^2 + 6x) - 2$

$= 2(x^2 + 6x + 9 - 9) - 2$

$= 2(x^2 + 6x + 9) - 18 - 2$

$= 2(x+3)^2 - 20$

b) $y = x^2 - 2x$

$= (x^2 - 2x + 1 - 1)$

$= (x^2 - 2x + 1) - 1$

$= (x-1)^2 - 1$

c) $y = -2x^2 - 12x - 23$

$= (-2x^2 - 12x) - 23$

$= -2(x^2 + 6x) - 23$

$= -2(x^2 + 6x + 9 - 9) - 23$

$= -2(x^2 + 6x + 9) + 18 - 23$

$= -2(x+3)^2 - 5$