1.1 - Functions, Domain, and Range

Lesson Outline

Section 1: Difference between relations and functions

Section 2: Use the vertical line test to check if it is a relation or a function

Section 3: Domain and range
Section 1: Relation vs. Function

Definitions

Relation: an identified pattern between two variables that may be represented as ordered pairs, a table of values, a graph, or an equation.

Function: a relation in which each value of the independent variable (the first coordinate) corresponds to exactly one value of the dependent variable (the second coordinate).

Note: All functions are relations but not all relations are functions. For a relation to be a function, there must be only one 'y' value that corresponds to a given 'x' value.
Function or Relation?

**Investigation**

1) Complete the following tables of values for each relation:

\[ y = x^2 \]

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<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 )</th>
<th>Coordinates</th>
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<td>3</td>
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\[ x = y^2 \]

<table>
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<th>( x = y^2 )</th>
<th>( y )</th>
<th>Coordinates</th>
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</table>

2) Graph both relations

\[ y = x^2 \]
3) Draw the vertical lines $x = -2, x = -1, x = 0, x = 1, \text{ and } x = 2$
on the graphs

$y = x^2$

$x = y^2$
4) Compare how the lines drawn in step 3 intersect each of the relations. Which relation is a function? Explain why.

None of the vertical lines drawn intersect the graph of \( y = x^2 \) at more than one point. That means each value of the independent variable corresponds to no more than one value of the dependent variable. So, the relation \( y = x^2 \) is a function.

The vertical lines drawn at \( x = 1 \), and \( x = 2 \) intersect the graph of \( x = y^2 \) at more than one point. That means some values of the independent variable corresponds to more than one value of the dependent variable. So, the relation \( x = y^2 \) is not a function.

Section 2: Vertical Line Test
**Vertical Line Test:** A method of determining whether a relation is a function or not. If every vertical line intersects the relation at only one point, then the relation is a function.

**Example 1:**

Use the vertical line test to determine whether each relation is a function or not.

**a)**

This relation is a function.

No vertical line can be drawn that passes through more than one point on the line.

**b)**

This relation is not a function.

An infinite number of vertical lines can be drawn that pass through more than one point on the curve. For example, $x = 6$ passes through $(6, 0)$ and $(6, 4)$.

**c)**

This relation is a function.
This relation is not a function.

Section 3: Domain and Range
For any relation, the set of values of the independent variable (often the $x$-values) is called the ______________ of the relation. The set of the corresponding values of the dependent variable (often the $y$-values) is called the ____________ of the relation.

**Note:** For a function, for each given element of the domain there must be exactly one element in the range.

**Domain:** values $x$ may take

**Range:** values $y$ may take

**General Notation**

**Domain:** $\{x \in \mathbb{R} \mid # \leq x \leq #\}$

**Domain:** $\{x = #, #, # \ldots \}$

**Range:** $\{y \in \mathbb{R} \mid # \leq y \leq #\}$

**Range:** $\{y = #, #, # \ldots \}$

**Real number:** a number in the set of all integers, terminating decimals, repeating decimals, non-terminating decimals, and non repeating decimals. Represented by the symbol $\mathbb{R}$
**Example 2:** Determine the domain and range of each relation from the data given.

a) \( \{ (-3, 4), (5, -3), (-2, 7), (5, 3), (6, -8) \} \)

\[
D: \{ x \in \mathbb{R} | x = -3, -2, 5, 6 \} \\
R: \{ y \in \mathbb{R} | y = -8, -3, 3, 7 \}
\]

b) 

<table>
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Are each of these relations functions?

a) is not a function. \( x = 5 \) has two different corresponding \( y \)-values.

b) is a function because for each element in the domain, there is exactly one corresponding value in the range. (no \( x \)-values repeated)

**Example 3:** Determine the domain and range of each relation. Graph the relation first. Then state if it is a function.

a) \( y = 2x - 5 \)

\[
D: \{ x \in \mathbb{R} \} \\
R: \{ y \in \mathbb{R} \}
\]
b) \[ y = (x - 1)^2 + 3 \]

vertex = \((h, k)\) = \((1, 3)\)

\[ D: \{ x \in \mathbb{R} \} \]

\[ R: \{ y \in \mathbb{R} \mid y \geq 3 \} \]

c) \[ y = \sqrt{x - 1} + 3 \]

\[ D: \{ x \in \mathbb{R} \mid x \geq 1 \} \]

\[ R: \{ y \in \mathbb{R} \mid y \geq 3 \} \]
Note:
vertical asymptotes effect the domain
horizontal asymptotes effect the range
Asymptotes

*Asymptote:* a line that a curve approaches more and more closely but never touches.

The function \( y = \frac{1}{x + 3} \) has two asymptotes:

**Vertical Asymptote:** Division by zero is undefined. Therefore the expression in the denominator of the function can not be zero. Therefore \( x \neq -3 \). This is why the vertical line \( x = -3 \) is an asymptote for this function.

**Horizontal Asymptote:** For the range, there can never be a situation where the result of the division is zero. Therefore the line \( y = 0 \) is a horizontal asymptote. For all functions where the denominator is a higher degree than the numerator, there will be a horizontal asymptote at \( y = 0 \).
Lesson Outline

Section 1: Do It Now

Section 2: 1.1 Review

Section 3: Find Values Using Function Notation

Section 4: Mapping Diagrams

Section 5: Application of Function Notation
DO IT NOW!

a) State the domain and range of the relation shown in the following graph:

\[ D: \{ x \in \mathbb{R} \mid -7 \leq x \leq 7 \} \]

\[ R: \{ y \in \mathbb{R} \mid -2.5 \leq y \leq 3 \} \]

b) Is this a function? **No!**

1.1 Review

What determines if a relation is a function or not?

A function is a relation in which each value of the independent variable \( x \) corresponds to exactly one value of the dependent variable \( y \).

How does the vertical line test help us determine if a relation is a function?

For any vertical line that is drawn through a relation, it should only cross at one point in order to be considered a function.
What is domain?

The set of values $x$ can take.

**Domain:** $\{x \in \mathbb{R} \mid \# \leq x \leq \#\}$

What is range?

The set of values $y$ can take.

**Range:** $\{y \in \mathbb{R} \mid \# \leq y \leq \#\}$

---

**Find Values Using Function Notation**

**What does a function do?**

Takes an input $(x)$, performs operations on it and then gives an output $(y)$

**What does function notation look like?**

$f(x) = \ldots$ something to do with $x$
Example 1: For each of the following functions, determine $f(2)$, $f(-5)$, and $f(1/2)$.

What is the value of the function (y-value) when $x = 2$?

a) $f(x) = 2x - 4$

\[ f(2) = 2(2) - 4 = 4 - 4 = 0 \]

Point on the line: $(2, 0)$

\[ f(-5) = 2(-5) - 4 = -10 - 4 = -14 \]

\[ f(\frac{1}{2}) = 2(\frac{1}{2}) - 4 = 1 - 4 = -3 \]

b) $f(x) = 3x^2 - x + 7$

\[ f(2) = 3(2)^2 - 2 + 7 = 3(4) - 2 + 7 = 12 - 2 + 7 = 17 \]

Point on the parabola: $(2, 17)$

\[ f(-5) = 3(-5)^2 - (-5) + 7 = 3(25) + 5 + 7 = 75 + 5 + 7 = 87 \]

\[ f(\frac{1}{2}) = 3(\frac{1}{2})^2 - \frac{1}{2} + 7 = 3(\frac{1}{4}) - \frac{1}{2} + 7 = \frac{3}{4} - \frac{1}{2} + 7 = \frac{3}{4} - \frac{2}{4} + \frac{28}{4} = \frac{31}{4} \text{ or } 7\frac{3}{4} \]

\[ f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 7 = 3\left(\frac{1}{4}\right) - \frac{1}{2} + 7 = \frac{3}{4} - \frac{1}{2} + 7 = \frac{3}{4} - \frac{2}{4} + \frac{28}{4} = \frac{31}{4} \text{ or } 7\frac{3}{4} \]
Note: $f(x) = 87$ is a constant function. It is a horizontal line through 87 on the y-axis.

d) $f(x) = \frac{2x}{x^2 - 3}$

$$f(2) = \frac{2(2)}{(2)^2 - 3} = \frac{4}{1} = 4$$  
$$(2, 4)$$

$$f(-5) = \frac{2(-5)}{(-5)^2 - 3} = \frac{-10}{-22} = \frac{5}{11}$$  
$$(-5, \frac{5}{11})$$

$$f\left(\frac{1}{3}\right) = \frac{2\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)^2 - 3} = \frac{1}{\frac{1}{9} - \frac{27}{9}} = \frac{1}{-\frac{26}{9}} = \frac{9}{-26} = -\frac{9}{26}$$  
$$\left(\frac{1}{3}, -\frac{9}{26}\right)$$
**Mapping Diagrams**

A mapping diagram is a representation that can be used when the relation is given as a set of ordered pairs. In a mapping diagram, the **domain** values in one oval are joined to the **range** values in the other oval using arrows.

A relation is a function if there is exactly one arrow leading from each value in the domain. This indicates that each element in the domain corresponds to exactly one element in the range.

**Example 2:** Use the mapping diagrams to

i) write the set of ordered pairs of the relation

ii) state if the relation is a function

Since every value in the domain maps to exactly one value in the range, this relation is a function.

i) \{(1, -5), (2, -1), (3, -5), (4, 0), (5, 0), (6, 5), (7, 8)\}
Since the values $x = 2$ and $x = 14$ both map to more than one value in the range, this relation is not a function.

\[ i) \{(2, 1), (2, 2), (5, 3), (8, 7), (11, 5), (14, 4), (14, 6)\} \]

**Applications of Function Notation**

**Example 3:** For the function \( h(t) = -3(t + 1)^2 + 5 \)

i) Graph it and find the domain and range

\[ \text{D:} \{x \in \mathbb{R}\} \]
\[ \text{R:} \{y \in \mathbb{R} \mid y \leq 5\} \]
ii) Find \( h(-7) \)

\[
h(t) = -3(t+1)^2 + 5
\]

\[
h(-7) = -3(-7+1)^2 + 5
\]

\[
= -3(36) + 5
\]

\[
= -108 + 5
\]

\[
= -103
\]

\((-7, -103)\)

Example 4:

The temperature of the water at the surface of a lake is 22 degrees Celsius. As Geno scuba dives to the depths of the lake, he finds that the temperature decreases by 1.5 degrees for every 8 meters he descends.

i) Model the water temperature at any depth using function notation.

\[
T(d) = \frac{-1.5}{8} d + 22
\]

**Hint:** it is a constant rate of change, therefore it will form a linear relation \((y=mx+b)\).
ii) What is the water temperature at a depth of 40 meters?

\[
T(40) = -\frac{1.5}{8} (40) + 22
\]

\[
= -1.5 (5) + 22
\]

\[
= -7.5 + 22
\]

\[
= 14.5 \, ^\circ C
\]

iii) At the bottom of the lake the temperature is 5.5 degrees Celsius. How deep is the lake?

\[
T(d) = -\frac{1.5}{8} d + 22
\]

\[
5.5 = -\frac{1.5}{8} d + 22
\]

\[
(8)(-16.5) = -\frac{1.5}{8} d (8)
\]

\[
-132 = -1.5 d
\]

\[
d = 88 \, \text{meters deep}.
\]
Lesson Outline

Section 1: Review of different forms of a quadratic equation

Section 2: Completing the square

Section 3: Partial Factoring

Section 5: Application
Quadratics Review

Vertex Form

\[ y = a(x - h)^2 + k \]

vertex = \((h, k)\)

cos: \(x = h\)

if \(a > 0\) ; opens up
so vertex is a min point

if \(a < 0\) ; opens down
so vertex is a max point

Factored Form

\[ y = a(x - r)(x - s) \]

The x-intercepts are \(r\) and \(s\)

\[ y = \frac{1}{2} (x+3)(x-2) \]
**Standard Form**

\[ y = ax^2 + bx + c \]

'c' is the y-intercept

use quadratic formula to find x-intercepts.

\[ x = \frac{-b}{2a} \] is the axis of symmetry and x-coordinate of the vertex.

**Perfect Square Trinomials**

\[(a + b)^2 = a^2 + 2ab + b^2\]

\[(a - b)^2 = a^2 - 2ab + b^2\]

Note:

- First and last terms are perfect squares
- Middle term = \(2\sqrt{\text{first term} \times \text{second term}}\)

**Factor each of the following:**

1) \(x^2 + 24x + 144\)

\[ = (x+12)^2 \]

2) \(x^2 - 18x + 81\)

\[ = (x-9)^2 \]
Can you make these perfect square trinomials?

\[ x^2 + 10x + 25 \]

\[ x^2 - 24x + 144 \]

The last term of a perfect square trinomial is half of the middle term squared!!!

and

the factored form is \((x + \frac{b}{2})^2\)

There are many ways to find the max or min of a quadratic function. The two methods taught in this lesson are:

1) completing the square

2) partial factoring
Completing the Square

Objective: Go from standard form to vertex form

\[ y = ax^2 + bx + c \quad \rightarrow \quad y = a(x-h)^2 + k \]

Once in vertex form, the max or min point is easy to find because the vertex is simply \((h, k)\)

\[ \begin{align*}
&\text{max if the parabola opens down } (a < 0) \\
&\text{min if the parabola opens up } (a > 0) \\
\end{align*} \]

Steps to Completing the Square:

1) Put brackets around the first two terms
2) Factor out the number in front of the \(x^2\)
3) Look at the last term in the brackets, divide it by 2 and then square it.
4) Add and subtract that term behind the last term in the brackets.
5) Move the negative term outside of the brackets by first multiplying it by the 'a' value.
6) Simplify the terms outside of the brackets.
7) Factor the perfect square trinomial
Example 1: Convert the following equation into vertex form (completing the square).

\[ y = x^2 + 8x + 5 \]

\[ y = (x^2 + 8x) + 5 \]
\[ y = (x^2 + 8x + 16 - 16) + 5 \]
\[ y = (x^2 + 8x + 16) - 16 + 5 \]
\[ y = (x + 4)^2 - 11 \]

What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?

vertex: \((h, k)\) \(-4, -11\)

a.o.s. \(x = h \rightarrow x = -4 \)

The vertex is a minimum point because the parabola opens up. \((a > 0)\)

Example 2: Convert the following equation into vertex form (completing the square).

\[ y = 2x^2 - 12x + 11 \]

\[ y = (2x^2 - 12x) + 11 \]
\[ y = 2(x^2 - 6x) + 11 \]
\[ y = 2(x^2 - 6x + 9 - 9) + 11 \]
\[ y = 2(x^2 - 6x + 9) - 18 + 11 \]
\[ y = 2(x - 3)^2 - 7 \]

What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?

vertex: \((h, k)\) \((3, -7)\)

a.o.s. \(x = h \rightarrow x = 3 \)

The vertex is a minimum point because the parabola opens up. \((a > 0)\)
Example 3: Convert the following equation into vertex form (completing the square).
\[
y = -3x^2 + 9x - 13
\]
\[
y = (-3x^2 + 9x) - 13
\]
\[
y = -3(x^2 - 3x) - 13
\]
\[
y = -3(x^2 - 3x + \frac{9}{4} - \frac{9}{4}) - 13
\]
\[
y = -3(x^2 - 3x + \frac{9}{4}) + \frac{27}{4} - \frac{65}{4}
\]
\[
y = -3\left(x - \frac{3}{2}\right)^2 - \frac{55}{4}
\]
What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?

vertex: \((h,k) = (\frac{3}{2}, -\frac{55}{4})
\)
axis of symmetry: \(x = \frac{3}{2}\)

The vertex is a maximum point because the parabola opens down (\(a < 0\)).

Example 4: Convert the following equation into vertex form (completing the square).
\[
y = -\frac{2}{3}x^2 + 8x + 5
\]
\[
y = \left(-\frac{2}{3}x^2 + 8x\right) + 5
\]
\[
y = -\frac{2}{3}(x^2 - 12x) + 5
\]
\[
y = -\frac{2}{3}(x^2 - 12x + 36 - 36) + 5
\]
\[
y = -\frac{2}{3}(x^2 - 12x + 36) + 34 + 5
\]
\[
y = -\frac{2}{3}(x - 6)^2 + 39
\]
What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?

vertex: \((h,k) = (6, 39)\) This is a max point.
axis of symmetry: \(x = h \rightarrow x = 6\)
Another Method to Find the Vertex

**PARTIAL FACTORING**

**Steps for Partial Factoring**

1. Look at the constant term in the expression. Set \( y = \) to the constant term.
2. Isolate the terms with a variable. Common factor what is left in the expression.
3. Solve the remaining equation for \( x \) by finding the two places where \( x \) is equal to zero.

**Note:** These are not the roots. These are just \( x \)-values that have to same \( y \)-values.

4. Find the axis of symmetry of the parabola by finding the average of the two \( x \)-values. You now know the \( x \)-value of the vertex.
5. Substitute the \( x \)-value of the vertex into the original equation and solve for the \( y \)-value.

**Example 5:** Use partial factoring to find the vertex. Then state if it is a max or min.

\[
y = x^2 + 2x - 6
\]

\[
-6 = x^2 + 2x - 6
\]

\[
-6 + 6 = x^2 + 2x
\]

\[
0 = x^2 + 2x
\]

\[
0 = x(x+2) \quad \text{(common factor)}
\]

\[
x = 0 \quad \text{or} \quad x+2=0 \quad \text{(zero product rule)}
\]

\[
x = -2
\]

The average of these two \( x \)-coordinates will give the \( x \)-coordinate of the vertex (a.o.s.)

\[
x_{\text{vertex}} = \frac{0+(-2)}{2} \quad y_{\text{vertex}} = (1)^2 + 2(-1) - 6
\]

\[
= -1 \quad \quad \quad = 1 - 2 - 6
\]

\[
= -7
\]

\( \text{So the vertex is } (-1, -7). \text{ It is a min.} \)
Example 6: Use partial factoring to find the vertex. Then state if it is a max or min.

\[ y = 4x^2 - 12x + 3 \]
\[ 3 = 4x^2 - 12x + 3 \]
\[ 3 - 3 = 4x^2 - 12x \]
\[ 0 = 4x^2 - 12x \]
\[ 0 = 4x(x - 3) \] (common factor)

\[ 4x = 0 \quad \text{or} \quad x - 3 = 0 \]
\[ x = 0 \quad \text{or} \quad x = 3 \]

The average of these x-coordinates will give the x-coordinate of the vertex (a.o.s.)

\[ x_{vertex} = \frac{0 + 3}{2} \]
\[ = \frac{3}{2} \]

\[ y_{vertex} = 4 \left( \frac{3}{2} \right)^2 - 12 \left( \frac{3}{2} \right) + 3 \]
\[ = 4 \left( \frac{9}{4} \right) - 6(3) + 3 \]
\[ = 9 - 18 + 3 \]
\[ = -6 \]

The vertex is \( \left( \frac{3}{2}, -6 \right) \). It is a min.

Example 7: Use partial factoring to find the vertex. Then state if it is a max or min.

\[ y = -3x^2 + 9x - 2 \]
\[ -2 = -3x^2 + 9x - 2 \]
\[ 0 = -3x^2 + 9x \]
\[ 0 = -3x(x - 3) \]

\[ -3x = 0 \quad \text{or} \quad x - 3 = 0 \]
\[ x = 0 \quad \text{or} \quad x = 3 \]

The average of these x-coordinates will give the x-coordinate of the vertex (a.o.s.)

\[ x_{vertex} = \frac{0 + 3}{2} \]
\[ = \frac{3}{2} \]

\[ y_{vertex} = -3 \left( \frac{3}{2} \right)^2 + 9 \left( \frac{3}{2} \right) - 2 \]
\[ = -3 \left( \frac{9}{4} \right) + \frac{27}{2} - 2 \]
\[ = -\frac{27}{4} + \frac{27}{2} - 2 \]
\[ = \frac{27}{4} - \frac{8}{4} \]
\[ = \frac{19}{4} \]

The vertex is \( \left( \frac{3}{2}, \frac{19}{4} \right) \). It is a maximum.
Example 8: Maximizing Profit (Application)

Rachel and Ken are knitting scarves to sell at the craft show. The wool for each scarf costs $6. They were planning to sell the scarves for $10 each, the same as last year when they sold 40 scarves. However, they know that if they raise the price, they will be able to make more profit, even if they end up selling fewer scarves. They have been told that for every 50c increase in the price, they can expect to sell four fewer scarves. What selling price will maximize their profit and what will the profit be?

**Step 1:** Find an equation to model their total profit.

Total Profit = (profit per scarf)(number of scarves sold)

\[
P(x) = (40 + 0.5x)(40 - 4x)
= 160 - 16x + 20x - 2x^2
= -2x^2 + 4x + 160
\]

**Step 2:** Find the max value of the function by either completing the square or by using partial factoring.

\[
P(x) = -2x^2 + 4x + 160
= (-2x^2 + 4x) + 160
= -2(x^2 - 2x) + 160
= -2(x^2 - 2x + 1 - 1) + 160
= -2((x - 1)^2) + 160 + 2 + 160
= -2(x - 1)^2 + 162
\]

The vertex is (1, 162) 

The max value of the quadratic function is 162 when $x = 1$. This means they will make a maximum profit of $162 if they increase the price once by $0.5. Therefore the selling price would be $10.50.
1.4 Working With Radicals

**Lesson Outline**

**Section 1:** Investigation

**Section 2:** Definitions

**Section 3:** Entire radicals to mixed radicals

**Section 4:** Add/Subtract radicals

**Section 5:** Multiply Radicals

**Section 6:** Application
Investigation

a) Complete the following table:

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<td>$\sqrt{31} \times \sqrt{31}$</td>
<td>31</td>
<td>$\sqrt{31} \times 31$</td>
</tr>
<tr>
<td>$\sqrt{12} \times \sqrt{9}$</td>
<td>10.39</td>
<td>$\sqrt{12} \times 9$</td>
</tr>
<tr>
<td>$\sqrt{23} \times \sqrt{121}$</td>
<td>52.75</td>
<td>$\sqrt{23} \times 121$</td>
</tr>
</tbody>
</table>

b) What do you notice about the results in each row?

The results are the same in each row.

c) Make a general conclusion about an equivalent expression for $\sqrt{a} \times \sqrt{b}$

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$
Definitions

Radicand:

a number or expression under a radical sign

Entire Radical:

a radical in the form $\sqrt{n}$, where $n > 0$, such as $\sqrt{45}$

Mixed Radical:

a radical in the form $a\sqrt{b}$, where $a \neq 1$ or -1 and $b > 0$, such as $3\sqrt{5}$

More About Radicals

Some numbers cannot be expressed as fractions. These are called __irrational____ numbers. One type of irrational number is of the form $\sqrt{n}$ where $n$ is not a perfect square. These numbers are sometimes called __radicals____.

An approximate value can be found for these irrational numbers using a calculator but it is better to work with an exact value. Answers should be left in radical form when an EXACT answer is needed. Sometimes entire radicals can be simplified by removing perfect square factors. The resulting expression is called a __mixed radical____________.
Example 1: Express each radical as a mixed radical in simplest form.

*Hint:* remove perfect square factors and then simplify

\[
\begin{align*}
\text{a)} \sqrt{50} \quad & \quad \text{b)} \sqrt{27} \quad & \quad \text{c)} \sqrt{180} \\
& = \sqrt{25 \times 2} & & = \sqrt{9 \times 3} & & = \sqrt{36 \times 5} \\
& = (\sqrt{25})(\sqrt{2}) & & = (\sqrt{9})(\sqrt{3}) & & = (\sqrt{36})(\sqrt{5}) \\
& = 5\sqrt{2} & & = 3\sqrt{3} & & = 6\sqrt{5}
\end{align*}
\]
Adding and Subtracting Radicals

Adding and subtracting radicals works in the same way as adding and subtracting polynomials. You can only add like terms or, in this case, like radicals.

Example:

\(2\sqrt{3} + 5\sqrt{7}\) cannot be added because they do not have the same radical.

However, \(3\sqrt{5} + 6\sqrt{5}\) have a common radical, so they can be added. \(3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}\)

Note, the radical stays the same when adding and subtracting expressions with like radicals.

**Example 2:** Simplify the following

a) \(9\sqrt{7} - 3\sqrt{7}\)

\[= 6\sqrt{7}\]

b) \(4\sqrt{3} - 2\sqrt{27}\)

\[= 4\sqrt{3} - 2(3)(\sqrt{3})\]

\[= 4\sqrt{3} - 6\sqrt{3}\]

\[= -2\sqrt{3}\]
Example 3:

Multiply the coefficients together and then multiply the radicands together. Then simplify!

\[ c) \ 5\sqrt{8} + 3\sqrt{18} \]

\[ = 5(2\sqrt{2})(\sqrt{2} \sqrt{3}) + 3(3\sqrt{2})(\sqrt{3}) \]

\[ = 5(2\sqrt{2}) + 3(3\sqrt{2}) \]

\[ = 10\sqrt{2} + 9\sqrt{2} \]

\[ = 19\sqrt{2} \]

\[ \text{d) } \frac{1}{4}\sqrt{28} - \frac{3}{4}\sqrt{63} + \frac{2}{3}\sqrt{50} \]

\[ = \frac{1}{4}(2\sqrt{7}) - \frac{3}{4}(3\sqrt{7}) + \frac{2}{3}(5\sqrt{2}) \]

\[ = \frac{2}{4}\sqrt{7} - \frac{9}{4}\sqrt{7} + \frac{10}{3}\sqrt{2} \]

\[ = \frac{-7}{4}\sqrt{7} + \frac{10}{3}\sqrt{2} \]

**Multiplying Radicals**

**Example 3:** Simplify fully

\[ \text{a) } (2\sqrt{3})(3\sqrt{6}) \]

\[ = (2)(3)(\sqrt{3})(\sqrt{6}) \]

\[ = 6\sqrt{18} \]

\[ = 6(\sqrt{9})(\sqrt{2}) \]

\[ = 6(3)(\sqrt{2}) \]

\[ = 18\sqrt{2} \]
b) \(2\sqrt{3}(4 + 5\sqrt{3})\)

\[= 2\sqrt{3}(4) + 2\sqrt{3}(5\sqrt{3})\]

\[= 8\sqrt{3} + 10\sqrt{9}\]

\[= 8\sqrt{3} + 10(3)\]

\[= 8\sqrt{3} + 30\]

Don't forget the distributive property:
\[a(x+y) = ax + ay\]

c) \(-7\sqrt{2}(6\sqrt{8} - 11)\)

\[= -7\sqrt{2}(6\sqrt{8}) - 7\sqrt{2}(-11)\]

\[= -42\sqrt{16} + 77\sqrt{2}\]

\[= -168 + 77\sqrt{2}\]
d) \((\sqrt{3} + 5)(2 - \sqrt{3})\)

Don't forget FOIL. Each term in the first binomial must be multiplied by each term in the second binomial.

\[= \sqrt{3}(2) + \sqrt{3}(-\sqrt{3}) + 5(2) + 5(-\sqrt{3})\]

\[= 2\sqrt{3} - 3 + 10 - 5\sqrt{3}\]

\[= 2\sqrt{3} - 5\sqrt{3} - 3 + 10\]

\[= -3\sqrt{3} + 7\]

e) \((2\sqrt{2} + 3\sqrt{3})(2\sqrt{2} - 3\sqrt{3})\)

There is a shortcut! This is a difference of squares.

\[(a + b)(a - b) = a^2 - b^2\]

\[= (2\sqrt{2})^2 - (3\sqrt{3})^2\]

\[= 4(2) - 9(3)\]

\[= 8 - 27\]

\[= -19\]
1.5 Solving Quadratic Equations

Part 1: Solve by Factoring

DO IT NOW!

1. Simplify. \( (\sqrt{2} + 3\sqrt{3})(5\sqrt{3} - 10) \)

\[ = \sqrt{2}(5\sqrt{3}) + \sqrt{2}(-10) + 3\sqrt{3}(5\sqrt{3}) + 3\sqrt{3}(-10) \]

\[ = 5\sqrt{6} - 10\sqrt{2} + 15\sqrt{9} - 30\sqrt{3} \]

\[ = 5\sqrt{6} - 10\sqrt{2} - 30\sqrt{3} + 45 \]
2. Simplify \( \frac{2 - \sqrt{80}}{4} \)

\[
= \frac{2 - \sqrt{16 \cdot 5}}{4} \\
= \frac{2 - 4\sqrt{5}}{4} \\
= \frac{2}{4} - \frac{4\sqrt{5}}{4} \\
= \frac{1}{2} - \sqrt{5}
\]

3. Expand and simplify:

\( 4\sqrt{10}(3 + 2\sqrt{2}) \)

\[
= 12\sqrt{10} + 8\sqrt{10} \cdot 2\sqrt{2} \\
= 12\sqrt{10} + 16\sqrt{20} \\
= 12\sqrt{10} + 16\sqrt{4 \cdot 5} \\
= 12\sqrt{10} + 16\cdot 2\sqrt{5} \\
= 12\sqrt{10} + 32\sqrt{5}
\]
Lesson Outline

Section 1: Solve a quadratic with an 'a' value of 1 or that can be factored out

Section 2: Solve a quadratic with an 'a' value of not 1 that can't be factored out.

*In all cases we will start with an equation in Standard Form and we will set it equal to 0:

\[ ax^2+bx+c = 0 \]

**NOTE:** If it's not in standard form, you must rearrange before factoring.

HOW TO SOLVE QUADRATICS

Solving a quadratic means to find the x-intercepts or roots.

To solve a quadratic equation:

1) It must be set to equal 0. Before factoring, it must be in the form \( ax^2+bx+c = 0 \)
2) Factor the left side of the equation
3) Set each factor to equal zero and solve for 'x'.

**zero product rule:** if two factors have a product of zero; one or both of the factors must equal zero.
**Example 1:** Solve the following quadratics by factoring

a) \( y = x^2 - 15x + 56 \)

\[
y = (x-8)(x-7)
\]

\[
0 = (x-8)(x-7)
\]

\[
x-8 = 0 \quad \text{or} \quad x-7 = 0
\]

\[
x = 8 \quad \quad x = 7
\]

b) \( y = x^2 - 5x + 6 \)

\[
0 = (x-2)(x-3)
\]

\[
0 = (x-2)(x-3)
\]

\[
x-2 = 0 \quad \text{or} \quad x-3 = 0
\]

\[
x = 2 \quad \quad x = 3
\]

c) \( y = 2x^2 - 8x - 42 \)

\[
0 = 2(x^2 - 4x - 21)
\]

\[
0 = 2(x-7)(x+3)
\]

\[
0 = (x-7)(x+3)
\]

\[
x-7 = 0 \quad \text{or} \quad x+3 = 0
\]

\[
x = 7 \quad \quad x = -3
\]

<table>
<thead>
<tr>
<th>When factoring ( ax^2+bx+c=0 ) when 'a' is 1 or can be factored out</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Steps to follow:</strong></td>
</tr>
<tr>
<td>1) Check if there is a common factor that can be divided out</td>
</tr>
<tr>
<td>2) Look at the 'c' value and the 'b' value</td>
</tr>
<tr>
<td>3) Determine what factors multiply to give 'c' and add to give 'b'</td>
</tr>
<tr>
<td>4) put those factors into ((x+r)(x+s)) for 'r' and 's'.</td>
</tr>
<tr>
<td>5) make sure nothing else can be factored</td>
</tr>
</tbody>
</table>
Steps to factoring $ax^2 + bx+c$ when 'a' cannot be factored out and is not 1.

1) Look to see if there is a common factor that can be divided out
2) Take the 'a' value and multiply it to the 'c' value
3) Determine what factors of THIS number add together to get the 'b' value
4) Break the 'b' value up into THOSE factors!
5) Put parenthesis around the first two variables and the last two
6) Factor by grouping

Example 2: Solve by factoring

a) $8x^2 + 2x - 15 = 0$

\[
\begin{align*}
8x^2 + 12x - 10x - 15 &= 0 \\
(8x^2 + 12x) + (-10x - 15) &= 0 \\
4x(2x+3) - 5(2x+3) &= 0 \\
(2x+3)(4x-5) &= 0 \\
2x+3 &= 0 \quad \text{or} \quad 4x-5 = 0 \\
x &= -\frac{3}{2} \quad \text{or} \quad x = \frac{5}{4}
\end{align*}
\]
Example 3: For the quadratic $y = 2x^2 - 4x - 16$

a) Find the roots of the quadratic by factoring

$$0 = 2(x^2 - 2x - 8)$$
$$0 = x^2 - 2x - 8$$
$$0 = (x - 4)(x + 2)$$

$x - 4 = 0$ or $x + 2 = 0$

$x = 4$ or $x = -2$

b) Find the axis of symmetry (average of x-intercepts)

$$a.o.s: \ x = \frac{r+s}{2} = \frac{4+(-2)}{2} = \frac{2}{2} = 1$$
e) Find the coordinates of the vertex and state if it is a max or min value

\[ x_{\text{vertex}} = 1 \]
\[ y_{\text{vertex}} = 2x^2 - 4x - 16 \]
\[ = 2(1)^2 - 4(1) - 16 \]
\[ = 2 - 4 - 16 \]
\[ = -18 \]

The vertex is \((1, -18)\).

This is a minimum value because the parabola opens up \((a > 0)\).
1.5 Solving Quadratic Equations

Part 2: Solve Using QF

Lesson Outline:

Part 1: Do It Now - QF Refresher

Part 2: Discriminant review

Part 3: Find exact solutions of a quadratic with 2 roots

Part 4: Solve a quadratic with 1 solution

Part 5: Solve a quadratic with 0 solutions

Part 6: Use the discriminant to determine the number of solutions (x-intercepts) a quadratic has

Part 7: Application
a) Do you remember the quadratic formula?

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

b) Use the quadratic formula to find the x-intercepts of:

$$0 = 2x^2 + 7x - 4$$

$$x = \frac{-7 \pm \sqrt{49 + 32}}{4}$$

$$x = \frac{-7 \pm \sqrt{81}}{4}$$

$$x = \frac{-7 + 9}{4} \text{ or } x = \frac{-7 - 9}{4}$$

$$x = \frac{1}{2} \text{ or } x = -4$$

---

**Part 2: Discriminant Review**

Do all parabolas have two x-intercepts?

**No**

What are the three different scenarios?

0, 1, or 2 solutions

The way to determine how many x-intercepts a parabola might have is by evaluating the $b^2 - 4ac$ part of the quadratic formula (called the "discriminant")

**Discriminant:** the value under the square root
Objective: Determine the roots of a quadratic using the quadratic formula and leave as EXACT answers. Exact answer: as a radical or fraction. Exact answers do not have decimals.

If \( b^2 - 4ac > 0 \), there are two solutions

If \( b^2 - 4ac = 0 \), there is one solution

If \( b^2 - 4ac < 0 \), there are no solutions

Part 3: Solve a Quadratic With 2 Roots
Example 1: Find the exact solutions of

\[ 3x^2 - 10x + 5 = 0 \]

\[ \chi = \frac{10 \pm \sqrt{(-10)^2 - 4 \cdot 3 \cdot 5}}{2 \cdot 3} \]

\[ = \frac{10 \pm \sqrt{100}}{6} \] (simplifying the radical)

\[ = \frac{10 \pm 10}{6} \]

\[ = \frac{10 \pm 10}{6} \] (common factor)

\[ = \frac{10(1 \pm 1)}{6} \]

\[ = \frac{10 \pm 10}{3} \] (reduced)

\[ \chi = \frac{5 + 10}{3} \quad \text{or} \quad \chi = \frac{5 - 10}{3} \]

Example 2: Find the exact solutions of

\[ -2x^2 + 8x - 5 = 0 \]

\[ \chi = \frac{-8 \pm \sqrt{(8)^2 - 4 \cdot (-2) \cdot (-5)}}{2 \cdot (-2)} \]

\[ = \frac{-8 \pm \sqrt{64}}{-4} \] (simplifying the radical)

\[ = \frac{-8 \pm 8 \sqrt{6}}{-4} \]

\[ = \frac{-8 \pm 8 \sqrt{6}}{-4} \] (common factor)

\[ = \frac{1}{2} \left( -4 \pm 4 \sqrt{6} \right) \]

\[ = \frac{-4 \pm 4 \sqrt{6}}{2} \] (reduced)

\[ \chi = \frac{-4 - 4 \sqrt{6}}{2} \quad \text{or} \quad \chi = \frac{-4 + 4 \sqrt{6}}{2} \]
Note: when a quadratic only has 1 solution, the x-intercept is also the vertex

Example 3: Find the exact roots of

\[ 4x^2 + 24x + 36 = 0 \]

\[ 4(x^2 + 6x + 9) = 0 \]
\[ x^2 + 6x + 9 = 0 \]

\[ x = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(9)}}{2(1)} \]
\[ x = \frac{-6 \pm \sqrt{36 - 36}}{2} \]
\[ x = \frac{-6 \pm 0}{2} \]
\[ x = \frac{-6}{2} \]
\[ x = -3 \]
2 Scenarios causing 0 roots:
i) vertex is above the x-axis and opens up
ii) vertex is below the x-axis and opens down

Example 4: Find the x-intercepts of

$$8x^2 - 11x + 5 = 0$$

$$\chi = \frac{11 \pm \sqrt{(11)^2 - 4(8)(5)}}{2(8)}$$

$$\chi = \frac{11 \pm \sqrt{-39}}{16}$$

& no solutions
### Part 6: Use the Discriminant to Determine the Number of Roots

**Example 5:** For each of the following quadratics, use the discriminant to state the number of roots it will have.

(a) \(2x^2 + 5x - 5 = 0\)

\[b^2 - 4ac = 5^2 - 4(2)(-5) = 25 + 40 = 60\]

\[60 > 0 \; \therefore 2 \text{ solutions}\]

(b) \(3x^2 - 7x + 5 = 0\)

\[b^2 - 4ac = (-7)^2 - 4(3)(5) = 49 - 60 = -11\]

\[-11 < 0 \; \therefore \text{no solutions}\]

(c) \(-4x^2 + 12x - 9 = 0\)

\[b^2 - 4ac = (12)^2 - 4(-4)(-9) = 144 - 144 = 0\]

\[0 = 0 \; \therefore 1 \text{ solution}\]

### Part 7: Application

**Example 6:** A ball is thrown and the equation below model it's path:

\[h = -0.25d^2 + 2d + 1.5\]

'h' is the height in meters above the ground and 'd' is the horizontal distance in meters from the person who threw the ball.

(a) At what height was the ball thrown from?

\[h = -0.25(0)^2 + 2(0) + 1.5\]

\[= 1.5 \text{ meters}\]
b) How far has the ball travelled horizontally when it lands on the ground?

\[ 0 = -0.25d^2 + 2d + 1.5 \]
\[ 0 = -0.25(d^2 - 8d - 6) \]
\[ 0 = d^2 - 8d - 6 \]

\[ d = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(-6)}}{2(1)} \]
\[ = \frac{8 \pm \sqrt{64 + 24}}{2} \]
\[ = \frac{8 \pm \sqrt{88}}{2} \]

\[ d = \frac{8 + \sqrt{88}}{2} \approx 8.7 \text{ meters} \]
\[ d = \frac{8 - \sqrt{88}}{2} \approx -0.7 \text{ meters} \]

Reject the negative solution.
1.7 Solve Linear-Quadratic Systems

Lesson Outline:

Part 1: Do It Now - review of substitution

Part 2: Possible solutions for a lin-quad system

Part 3: Solve linear-quadratic systems

Part 4: Application
DO IT NOW!

Solve the following linear system using the method of substitution:

\[ y = 3x + 7 \]
\[ y = 2x - 5 \]

Recall: solving a linear system means to find the point of intersection (POI)

Method of Substitution: solving a linear system by substituting for one variable from one equation into the other equation.

\[ 3x + 7 = 2x - 5 \]
\[ 3x - 2x = -5 - 7 \]
\[ x = -12 \]

Sub x-value back in to \( 1 \) or \( 2 \) and solve for y:

\[ y = 3(-12) + 7 \]
\[ y = -29 \]

so the POI is \((-12, -29)\)
**Possible solutions for a linear-quadratic system:**

<table>
<thead>
<tr>
<th>2 intersections</th>
<th>1 intersection</th>
<th>0 intersections</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
<td><img src="image3" alt="Graph" /></td>
</tr>
</tbody>
</table>

- **Secant**: A line that intersects a curve at two distinct points.
- **Tangent line**: A line that touches a curve at one point and has the slope of the curve at that point.
- **Discriminant**:
  - $\text{discriminant} > 0$
  - $\text{discriminant} = 0$

---

**Example 1**

a) How many points of intersection are there for the following system of equations?

\[
f(x) = \frac{1}{2}x^2 + 2x - 8 \quad g(x) = 4x - 10
\]

Set $f(x) = g(x)$

\[
\frac{1}{2}x^2 + 2x - 8 = 4x - 10
\]

(set equal to each other)

\[
\frac{1}{2}x^2 + 2x - 8 + 10 = 0
\]

(set equal to zero)

\[
\frac{1}{2}x^2 - 2x + 2 = 0
\]

(common factor)

\[
\frac{1}{2}(x^2 - 4x + 4) = 0
\]

\[
x^2 - 4x + 4 = 0
\]

$\quad b^2 - 4ac = (-4)^2 - 4(1)(4) \quad \text{(check discriminant)}$

\[
= 0
\]

& 1 solution
b) Solve the linear-quadratic system (give exact answers)

\[ x^2 - 4x + 4 = 0 \]
\[ (x-2)^2 = 0 \]
\[ x = 2 \]

Plug \( x = 2 \) back in to either original equation (linear is usually easier)

\[ g(x) = 4x-10 \]
\[ g(2) = 4(2)-10 = -2 \]

So, the POI is \((2, -2)\)

---

**Example 2**

Solve the following linear quadratic system

\[ y = 3x^2 + 21x - 5 \]
\[ y = 10x - 1 \]

\[ 3x^2 + 21x - 5 = 10x - 1 \]
\[ 3x^2 + 11x - 4 = 0 \]
\[ (x+4)(3x-1) = 0 \]
\[ x+4 = 0 \quad 3x-1 = 0 \]
\[ x_1 = -4 \quad x_2 = \frac{1}{3} \]

\[ \text{POI #1} \]
\[ y = 10(-4) - 1 = -41 \]
\[ y = -41 \]
\[ (-4, -41) \]

\[ \text{POI #2} \]
\[ y = 10\left(\frac{1}{3}\right) - 1 = \frac{10}{3} - \frac{3}{3} = \frac{7}{3} \]
\[ y = \frac{7}{3} \]
\[ \left(\frac{1}{3}, \frac{7}{3}\right) \]
Part 4: Application

Example 3: If a line with slope 4 has one point of intersection with the quadratic function \( y = \frac{1}{2}x^2 + 2x - 8 \), what is the \( y \)-intercept of the line? Write the equation of the line in slope \( y \)-intercept form.

\[
y = 4x + k
\]

Recall: equation of a line is \( y = mx + k \) where \( k \) is the \( y \)-intercept and \( m \) is the slope.

Recall: for a lin-quad system to have 1 solution, the discriminant must be zero.

\[
4x + k = \frac{1}{2}x^2 + 2x - 8 \\
0 = \frac{1}{2}x^2 - 2x - 8 - k
\]

Then \( a = \frac{1}{2}, \ b = -2, \) and \( c = -8 - k \)

\[
b^2 - 4ac = 0 \\
(-2)^2 - 4(\frac{1}{2})(-8 - k) = 0 \\
4 - 2(-8 - k) = 0 \\
4 + 16 + 2k = 0 \\
2k = -20 \\
k = -10
\]

So the equation of the line must be \( y = 4x - 10 \)