Chapter 1- Functions

Lesson Package

MCR3U
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1.1 Functions, Domain, and Range - Lesson
MCR3U
Jensen

Section 1: Relation vs. Function

Definitions

Relation -

Functions -

Note: All functions are relations but not all relations are functions. For a relation to be a function, there must be only one 'y' value that corresponds to a given 'x' value.

Function or Relation Investigation

1) Complete the following tables of values for each relation:

\[ y = x^2 \] \hspace{1cm} \[ x = y^2 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 )</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
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<td>-3</td>
<td>9</td>
<td>(-3, 9)</td>
</tr>
<tr>
<td>-2</td>
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<td>0</td>
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<tr>
<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>\</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>( x = y^2 )</th>
<th>( y )</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>-3</td>
<td>(9, -3)</td>
</tr>
<tr>
<td>\</td>
<td>-2</td>
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<tr>
<td>\</td>
<td>3</td>
<td>\</td>
</tr>
</tbody>
</table>
2) Graph both relations

\[ y = x^2 \]

\[ x = y^2 \]

3) Draw the vertical lines \( x = -2, x = -1, x = 0, x = 1, \) and \( x = 2 \) on the graphs above.

4) Compare how the lines drawn in step 3 intersect each of the relations. Which relation is a function? Explain why.

**Section 2: Vertical Line Test**

**Vertical line test:**

**Example 1:** Use the vertical line test to determine whether each relation is a function or not.

a) 

b)
Section 3: Domain and Range

For any relation, the set of values of the independent variable (often the x-values) is called the ______________ of the relation. The set of the corresponding values of the dependent variable (often the y-values) is called the ____________ of the relation.

Note: For a function, for each given element of the domain there must be exactly one element in the range.

General Notation

Real number: a number in the set of all integers, terminating decimals, repeating decimals, non-terminating decimals, and non repeating decimals. Represented by the symbol \( \mathbb{R} \)
Example 2: Determine the domain and range of each relation from the data given.

a) \{ (-3, 4), (5, -6), (-2, 7), (5, 3), (6, -8) \}

b) Are each of these relations functions?

Example 3: Determine the domain and range of each relation. Graph the relation first.

a) \( y = 2x - 5 \)
b)  \( y = (x - 1)^2 + 3 \)

c)  \( y = \sqrt{x - 1} + 3 \)

d)  \( x^2 + y^2 = 36 \)
Asymptotes

Asymptote:

The function \( y = \frac{1}{x+3} \) has two asymptotes:

**Vertical Asymptote:** Division by zero is undefined. Therefore the expression in the denominator of the function can not be zero. Therefore \( x \neq -3 \). This is why the vertical line \( x = -3 \) is an asymptote for this function.

**Horizontal Asymptote:** For the range, there can never be a situation where the result of the division is zero. Therefore the line \( y = 0 \) is a horizontal asymptote. For all functions where the denominator is a higher degree than the numerator, there will by a horizontal asymptote at \( y = 0 \).
DO IT NOW!

a) State the domain and range of the relation shown in the following graph:

b) Is this a function?

1.1 Review

What determines if a relation is a function or not?

How does the vertical line test help us determine if a relation is a function?

What is domain?

What is range?
Find Values Using Function Notation

What does a function do?

What does function notation look like?

**Example 1:** For each of the following functions, determine \( f(2) \), \( f(-5) \), and \( f(1/2) \)

a) \( f(x) = 2x - 4 \)

b) \( f(x) = 3x^2 - x + 7 \)

c) \( f(x) = 87 \)

d) \( f(x) = \frac{2x}{x^2 - 3} \)
Mapping Diagrams

A mapping diagram is a representation that can be used when the relation is given as a set of ordered pairs. In a mapping diagram, the _______ values in one oval are joined to the _______ values in the other oval using arrows.

A relation is a function if there is exactly ______ arrow leading from each value in the domain. This indicates that each element in the domain corresponds to exactly one element in the range.

Example 2: Use the mapping diagrams to

i) write the set of ordered pairs of the relation
ii) state if the relation is a function

a)

b)
Applications of Function Notation

Example 3: For the function \( h(t) = -3(t + 1)^2 + 5 \)

i) Graph it and find the domain and range

ii) Find \( h(-7) \)

Example 4: The temperature of the water at the surface of a lake is 22 degrees Celsius. As Geno scuba dives to the depths of the lake, he finds that the temperature decreases by 1.5 degrees for every 8 meters he descends.

i) Model the water temperature at any depth using function notation.

\[ y = -1.5x + 22 \] (where \( x \) is the depth in meters and \( y \) is the temperature in degrees Celsius)

Hint: it is a constant rate of change, therefore it will form a linear relation \( (y=mx+b) \).

ii) What is the water temperature at a depth of 40 meters?

iii) At the bottom of the lake the temperature is 5.5 degrees Celsius. How deep is the lake?
1.3 Max or Min of a Quadratic Function – Lesson
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Quadratics Review

Vertex Form:

\[ y = a(x - h)^2 + k \]

Factored Form:

\[ y = a(x - r)(x - s) \]
Standard Form:

\[ y = ax^2 + bx + c \]

Perfect Square Trinomials

Note:

- First and last terms are perfect squares
- Middle term = \( 2\sqrt{\text{first term} \cdot \text{second term}} \)

Factor each of the following:

1) \( x^2 + 24x + 144 \)

2) \( x^2 - 18x + 81 \)

Can you make these perfect square trinomials?

\[ x^2 + 10x + \_ \_ \_ \]

\[ x^2 - 24x + \_ \_ \_ \]

*The last term of a perfect square trinomial is half of the middle term squared!!!

and the factored form is \( (x + \frac{b}{2})^2 \)
There are many ways to find the max or min of a quadratic function. The two methods taught in this lesson are:

1) completing the square

2) partial factoring

**Completing the Square**

**Objective:** Go from standard form to vertex form

\[ y = ax^2 + bx + c \quad \rightarrow \quad y = a(x-h)^2 + k \]

Once in vertex form, the max or min point is easy to find because the vertex is simply \((h, k)\)

---

**Steps to Competing the Square:**

1) Put brackets around the first two terms
2) Factor out the number in front of the \(x^2\)
3) Look at the last term in the brackets, divide it by 2 and then square it.
4) Add and subtract that term behind the last term in the brackets.
5) Move the negative term outside of the brackets by first multiplying it by the 'a' value.
6) Simplify the terms outside of the brackets.
7) Factor the perfect square trinomial

---

**Example 1:** Convert the following equation into vertex form (completing the square).

\[ y = x^2 + 8x + 5 \]

What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?
Example 2: Convert the following equation into vertex form (completing the square).

\[ y = 2x^2 - 12x + 11 \]

What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?

Example 3: Convert the following equation into vertex form (completing the square).

\[ y = -3x^2 + 9x - 13 \]

What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?
Example 4: Convert the following equation into vertex form (completing the square).

\[ y = -\frac{2}{3}x^2 + 8x + 5 \]

What is the vertex and axis of symmetry of the parabola? Is the vertex a max or min point?

Partial Factoring (another method to find the vertex)

**Steps for Partial Factoring**

1. Look at the constant term in the expression. Set \( y = \) to the constant term.
2. Isolate the terms with a variable. Common factor what is left in the expression.
3. Solve the remaining equation for \( x \) by finding the two places where \( x \) is equal to zero.

**Note:** These are not the roots. These are just \( x \)-values that have to same \( y \)-values.

4. Find the axis of symmetry of the parabola by finding the average of the two \( x \)-values. You now know the \( x \)-value of the vertex.
5. Substitute the \( x \)-value of the vertex into the original equation and solve for the \( y \)-value.
**Example 5:** Use partial factoring to find the vertex. Then state if it is a max or min.

\[ y = x^2 + 2x - 6 \]

**Example 6:** Use partial factoring to find the vertex. Then state if it is a max or min.

\[ y = 4x^2 - 12x + 3 \]

**Example 7:** Use partial factoring to find the vertex. Then state if it is a max or min.

\[ y = -3x^2 + 9x - 2 \]
Example 8: Maximizing Profit (application)

Rachel and Ken are knitting scarves to sell at the craft show. The wool for each scarf costs $6. They were planning to sell the scarves for $10 each, the same as last year when they sold 40 scarves. However, they know that if they raise the price, they will be able to make more profit, even if they end up selling fewer scarves. They have been told that for every 50¢ increase in the price, they can expect to sell four fewer scarves. What selling price will maximize their profit and what will the profit be?

Step 1: Find an equation to model their total profit.

Step 2: Find the max value of the function by either completing the square or by using partial factoring.

Key Concepts

- The minimum or maximum value of a quadratic function occurs at the vertex of the parabola.
- The vertex of a quadratic function can be found by
  - graphing
  - completing the square: for \( f(x) = a(x - h)^2 + k \), the vertex is \((h, k)\)
  - partial factoring: for \( f(x) = a(x + \frac{b}{a})^2 + k \), the x-coordinate of the vertex is \(-\frac{b}{2a}\)
- The sign of the coefficient \(a\) in the quadratic function \( f(x) = ax^2 + bx + c \) or \( f(x) = a(x - h)^2 + k \) determines whether the vertex is a minimum or a maximum.
  - If \( a > 0 \), then the parabola opens upward and has a minimum.
  - If \( a < 0 \), then the parabola opens downward and has a maximum.
Investigation

a) Complete the following table:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{4} \times \sqrt{4} = $</td>
<td>$\sqrt{4} \times \sqrt{4} = $</td>
</tr>
<tr>
<td>$\sqrt{81} \times \sqrt{81} = $</td>
<td>$\sqrt{81} \times \sqrt{81} = $</td>
</tr>
<tr>
<td>$\sqrt{225} \times \sqrt{225} = $</td>
<td>$\sqrt{225} \times \sqrt{225} = $</td>
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<tr>
<td>$\sqrt{5} \times \sqrt{5} = $</td>
<td>$\sqrt{5} \times \sqrt{5} = $</td>
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<td>$\sqrt{31} \times \sqrt{31} = $</td>
<td>$\sqrt{31} \times \sqrt{31} = $</td>
</tr>
<tr>
<td>$\sqrt{12} \times \sqrt{9} = $</td>
<td>$\sqrt{12} \times \sqrt{9} = $</td>
</tr>
<tr>
<td>$\sqrt{23} \times \sqrt{121} = $</td>
<td>$\sqrt{23} \times \sqrt{121} = $</td>
</tr>
</tbody>
</table>

b) What do you notice about the results in each row?

c) Make a general conclusion about an equivalent expression for $\sqrt{a} \times \sqrt{b}$

Definitions

**Radicand:**

**Entire Radical:**

**Mixed Radical:**
More About Radicals

Some numbers cannot be expressed as fractions. These are called ______________ numbers. One type of irrational number is of the form $\sqrt{n}$ where $n$ is not a perfect square. These numbers are sometimes called ____________.

An approximate value can be found for these irrational numbers using a calculator but it is better to work with an exact value. Answers should be left in radical form when an EXACT answer is needed. Sometimes entire radicals can be simplified by removing perfect square factors. The resulting expression is called a __________________________.

![Perfect Squares Table]

**Example 1:** Express each radical as a mixed radical in simplest form.

*Hint: remove perfect square factors and then simplify*

a) $\sqrt{50}$  
b) $\sqrt{27}$  
c) $\sqrt{180}$
Adding and Subtracting Radicals

Adding and subtracting radicals works in the same way as adding and subtracting polynomials. You can only add ________ terms or, in this case, __________ ________________.

Example:

$2\sqrt{3} + 5\sqrt{7}$ cannot be added because they do not have the same radical.

However, $3\sqrt{5} + 6\sqrt{5}$ have a common radical, so they can be added. $3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}$

Note, the radical stays the same when adding and subtracting expressions with like radicals.

Example 2: Simplify the following

a) $9\sqrt{7} - 3\sqrt{7}$

b) $4\sqrt{3} - 2\sqrt{27}$

c) $5\sqrt{8} + 3\sqrt{18}$

d) $\frac{1}{4}\sqrt{28} - \frac{3}{4}\sqrt{63} + \frac{2}{3}\sqrt{50}$
### Multiplying Radicals

**Example 3:** Simplify fully

a) \((2\sqrt{3})(3\sqrt{6})\)

Multiply the coefficients together and then multiply the radicands together. Then simplify!

b) \(2\sqrt{3}(4 + 5\sqrt{3})\)

Don't forget the distributive property:
\[a(x+y) = ax + ay\]

c) \(-7\sqrt{2}(6\sqrt{8} - 11)\)

d) \((\sqrt{3} + 5)(2 - \sqrt{3})\)

Don't forget FOIL. Each term in the first binomial must be multiplied by each term in the second binomial.

e) \((2\sqrt{2} + 3\sqrt{3})(2\sqrt{2} - 3\sqrt{3})\)

There is a shortcut! This is a difference of squares.
\[(a + b)(a - b) = a^2 - b^2\]
DO IT NOW

Simplify each of the following:

1) \((\sqrt{2} + 3\sqrt{3})(5\sqrt{3} - 10)\)

2) \(\frac{2 - \sqrt{80}}{4}\)

3) \(4\sqrt{10}(3 + 2\sqrt{2})\)

*In all cases we will start with an equation in Standard Form and we will set it equal to 0:

\[ax^2 + bx + c = 0\]

**NOTE:** If it's not in standard form, you must rearrange before factoring.

How to Solve Quadratics

Solving a quadratic means to find the x-intercepts (or roots).

To solve a quadratic equation:

1) It must be set to equal 0. Before factoring, it must be in the form \(ax^2 + bx + c = 0\)
2) Factor the left side of the equation
3) Set each factor to equal zero and solve for 'x'.

**zero product rule:** if two factors have a product of zero; one or both of the factors must equal zero.
Example 1: Solve the following quadratics by factoring

a) \( y = x^2 - 15x + 56 \)

When factoring \( ax^2+bx+c=0 \) when 'a' is 1 or can be factored out

Steps to follow:
1) Check if there is a common factor that can be divided out
2) Look at the 'c' value and the 'b' value
3) Determine what factors multiply to give 'c' and add to give 'b'
4) put those factors into \((x+r)(x+s)\) for 'r' and 's'.
5) make sure nothing else can be factored

\[ b) \ y = x^2 - 5x + 6 \]
\[ c) \ y = 2x^2 - 8x - 42 \]

Example 2: Solve by factoring

a) \( 8x^2 + 2x - 15 = 0 \)

Steps to factoring \( ax^2 +bx+c \) when 'a' cannot be factored out and is not 1.

1) Look to see if there is a common factor that can be divided out
2) Take the 'a' value and multiply it to the 'c' value
3) Determine what factors of THIS number add together to get the 'b' value
4) Break the 'b' value up into THOSE factors!
5) Put parenthesis around the first two variables and the last two
6) Factor by grouping
b) \(2x^2 - 11x = -15\)

**Example 3:** For the quadratic \(y = 2x^2 - 4x - 16\)

a) Find the roots of the quadratic by factoring

b) Find the axis of symmetry (average of x-intercepts)

c) Find the coordinates of the vertex and state if it is a max or min value
1.5 Solving Quadratic Equations – Part 2: Solve Using the Q.F.

Part 1: DO IT NOW!

a) Do you remember the quadratic formula?

b) Use the quadratic formula to find the x-intercepts of:

\[ 0 = 2x^2 + 7x - 4 \]

Don’t forget that to solve a quadratic, it must be set equal to zero because at an x-intercept, the y-coordinate will be zero.

Part 2: Discriminant Review

Do all parabolas have two x-intercepts?

What are the three different scenarios?

The way to determine how many x-intercepts a parabola might have is by evaluating the \( b^2 - 4ac \) part of the quadratic formula (called the "discriminant")

**Discriminant:** the value under the square root

- **If** \( b^2 - 4ac > 0 \), **there are two solutions**
- **If** \( b^2 - 4ac = 0 \), **there is one solution**
- **If** \( b^2 - 4ac < 0 \), **there are no solutions**
Part 3: Solve a Quadratic with 2 Roots

Objective: Determine the roots of a quadratic using the quadratic formula and leave as EXACT answers

Exact answer: as a radical or fraction. Exact answers do not have decimals.

Example 1: Find the exact solutions of $3x^2 - 10x + 5 = 0$

Example 2: Find the exact solutions of $-2x^2 + 8x - 5 = 0$

Part 4: Solving a Quadratic with 1 Root

Note: when a quadratic only has 1 solution, the x-intercept is also the vertex

Example 3: Find the exact roots of $4x^2 + 24x + 36 = 0$
2 Scenarios causing 0 roots:
   i) vertex is above the x-axis and opens up
   ii) vertex is below the x-axis and opens down

Example 4: Find the x-intercepts of \( 8x^2 - 11x + 5 = 0 \)

Part 6: Use the Discriminant to Determine the Number of Roots

Example 5: For each of the following quadratics, use the discriminant to state the number of roots it will have.

a) \( 2x^2 + 5x - 5 = 0 \)

b) \( 3x^2 - 7x + 5 = 0 \)

c) \( -4x^2 + 12x - 9 = 0 \)
Part 7: Application

Example 6: A ball is thrown and the equation below model its path:

\[ h = -0.25d^2 + 2d + 1.5 \]

'h' is the height in meters above the ground and 'd' is the horizontal distance in meters from the person who threw the ball.

a) At what height was the ball thrown from?

b) How far has the ball travelled horizontally when it lands on the ground?
DO IT NOW!

Use the quadratic formula to find the exact roots of $2x^2 - 4x - 3 = 0$

Part 2: Review of Factored Form Equation

RECALL: the equation of a parabola in factored form is

$$y = a(x - r)(x - s)$$

where $r$ and $s$ are the x intercepts (zeros)

How to write the equation of a parabola in standard form given the x-intercepts and a point:

1) Plug values for $r$ and $s$ into $y = a(x - r)(x - s)$. Then simplify if possible.
2) Plug values for $x$ and $y$ into $y = a(x - r)(x - s)$.
3) Solve for $a$
4) Write the equation in factored form by plugging in values for $a, r, and s$. Not $x$ and $y$.
5) Expand the equation by multiplying the factors to convert to standard form ($y = ax^2 + bx + c$).
Part 3: Determine the Equation of a Quadratic Given Roots and a Point

Example 1: Determine the exact equation of a quadratic function with zeros at 2 and -2, containing the point (0, 3).

Example 2: Determine the exact equation of a quadratic function with double zero at -2, containing the point (3, 10).

Example 3: Determine the exact equation of a quadratic function with zeros at $3 + \sqrt{5}$ and $3 - \sqrt{5}$, containing the point (2, -12).
Part 4: Application

Example 4: Bridges like the one shown often have supports in the shape of parabolas. If the anchors at either side of the bridge are 42 meters apart and the maximum height of the support is 26 meters, what function models the parabolic curve of the support?

Note: we must assume the vertex is on the y-axis and the x-intercepts are on the x-axis.
Solve the following linear system using the method of substitution:

\[ y = 3x + 7 \]
\[ y = 2x - 5 \]

**Recall:** solving a linear system means to find the point of intersection (POI)

**Method of Substitution:** solving a linear system by substituting for one variable from one equation into the other equation.

**Steps to Solving A Linear-Quadratic System**

1. Set equations equal to each-other
   
   Line = Parabola

2. Rearrange to set the equation equal to zero

3. Solve for \( x \) by factoring or using the QF (the solution will tell you for what value of \( x \) the functions have the same \( y \) value)

4. Plug this value of \( x \) back in to either of the original functions to solve for \( y \).
Possible solutions for a linear-quadratic system:

<table>
<thead>
<tr>
<th>2 intersections</th>
<th>1 intersection</th>
<th>0 intersections</th>
</tr>
</thead>
</table>

**Secant:** A line that intersects a curve at two distinct points.

**Tangent line:** A line that touches a curve at one point and has the slope of the curve at that point.

- discriminant > 0
- discriminant = 0

Example 1:

**a)** How many points of intersection are there for the following system of equations?

\[
f(x) = \frac{1}{2} x^2 + 2x - 8 \quad g(x) = 4x - 10
\]
b) Solve the linear-quadratic system (give exact answers)

**Example 2:** Solve the following linear-quadratic system

\[ y = 3x^2 + 21x - 5 \]
\[ y = 10x - 1 \]
Part 4: Application

Example 3: If a line with slope 4 has one point of intersection with the quadratic function 
\( y = \frac{1}{2} x^2 + 2x - 8 \), what is the y-intercept of the line? Write the equation of the line in slope y-intercept form.

\[
\begin{align*}
\text{Recall: equation of a line is } & \quad y = mx + b \text{ where } b \text{ is the y-intercept and } m \text{ is the slope.} \\
\text{Recall: for a lin-quad system to have 1 solution, the discriminant must be zero.}
\end{align*}
\]