Chapter 5 Exam Review

SOLUTIONS

MDM4U
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1. The door prizes at a dance are gift certificates from local merchants. There are four $10 certificates, five $20 certificates, and three $50 certificates. The prize envelopes are mixed together in a bag and are drawn at random.

a) Create a probability distribution for the number of $50 prizes drawn, n, on the first three draws.

# of $50 prizes drawn ($n$)	$P(n)$
0	$\frac{\binom{3}{0}\binom{9}{3}}{\binom{12}{3}} = \frac{84}{220} = 0.3818$
1	$\frac{\binom{3}{1}\binom{9}{2}}{\binom{12}{3}} = \frac{108}{220} = 0.4909$
2	$\frac{\binom{3}{2}\binom{9}{1}}{\binom{12}{3}} = \frac{27}{220} = 0.1227$
3	$\frac{\binom{3}{3}\binom{9}{0}}{\binom{12}{3}} = \frac{1}{220} = 0.0045$

b) What is the expected number of $50 certificates among the first three prizes drawn?

$$E(n) = 0\left(\frac{84}{220}\right) + 1\left(\frac{108}{220}\right) + 2\left(\frac{27}{220}\right) + 3\left(\frac{1}{220}\right)$$

$$= \frac{108}{220} + \frac{54}{220} + \frac{3}{220}$$

$$= \frac{165}{220}$$

$$= 0.75$$

2. A family plans on having four children. Assuming the probability of having a boy is equal to the probability of having a girl...

a) Create a probability distribution for the number of boys, X, the family will have

# of boys (X)	$P(X)$
0	$\binom{4}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^4 = \frac{1}{16}$
1	$\binom{4}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 = \frac{4}{16}$
2	$\binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{6}{16}$
3	$\binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 = \frac{4}{16}$
4	$\binom{4}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0 = \frac{1}{16}$

b) Find the expected number of boys in a family with four children

$$E(X) = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right)$$

$$= \frac{4}{16} + \frac{12}{16} + \frac{12}{16} + \frac{4}{16}$$

$$= \frac{32}{16}$$

$$= 2$$

3. A game is designed in which you roll two die and the sum is noted, if you roll doubles you win $100, if you roll a sum of 5 you win $25, and on all else you pay $5

a) Complete a probability distribution table for the amount you win/lose if you play the game.

Amount Won/Lost (X)	$P(X)$
$100	$\frac{6}{36} = \frac{1}{6}$
$25	$\frac{4}{36} = \frac{1}{9}$
$ -5	$\frac{26}{36} = \frac{13}{18}$

(1,4)(4,1)(2,3)(3,2)

b) What is the amount that you would expect to win/lose if you played the game 10 times?

$$E(X) = 100\left(\frac{6}{36}\right) + 25\left(\frac{4}{36}\right) - 5\left(\frac{26}{36}\right)$$

$$= \frac{600}{36} + \frac{100}{36} - \frac{130}{36}$$

$$= \frac{570}{36}$$

$$= \$15.83$$

You would expect to win $15.83 × 10 = $158.30 if you play 10 times.

4. Find the binomial expansion of each expression in simplified form.

a) $(2x + 3)^4$ $= \binom{4}{0}(2x)^4(3)^0 + \binom{4}{1}(2x)^3(3)^1 + \binom{4}{2}(2x)^2(3)^2 + \binom{4}{3}(2x)^1(3)^3 + \binom{4}{4}(2x)^0(3)^4$

$= 16x^4 + 96x^3 + 216x^2 + 216x + 81$

b) $(2x-1)^4$ $= \binom{4}{0}(2x)^4(-1)^0 + \binom{4}{1}(2x)^3(-1)^1 + \binom{4}{2}(2x)^2(-1)^2 + \binom{4}{3}(2x)^1(-1)^3 + \binom{4}{4}(2x)^0(-1)^4$

$= 16x^4 - 32x^3 + 24x^2 - 8x + 1$

c) $(3x-2y)^5$

$= \binom{5}{0}(3x)^5(-2y)^0 + \binom{5}{1}(3x)^4(-2y)^1 + \binom{5}{2}(3x)^3(-2y)^2 + \binom{5}{3}(3x)^2(-2y)^3 + \binom{5}{4}(3x)^1(-2y)^4 + \binom{5}{5}(3x)^0(-2y)^5$

$= 243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$

d) $\left(x+\frac{2}{x^2}\right)^4$

$= \binom{4}{0}(x)^4\left(\frac{2}{x^2}\right)^0 + \binom{4}{1}(x)^3\left(\frac{2}{x^2}\right)^1 + \binom{4}{2}(x)^2\left(\frac{2}{x^2}\right)^2 + \binom{4}{3}(x)^1\left(\frac{2}{x^2}\right)^3 + \binom{4}{4}(x)^0\left(\frac{2}{x^2}\right)^4$

$= x^4 + 8x + 24x^{-2} + 32x^{-5} + 16x^{-8}$

5. Find the 8th term in the expansion of $(x^3 - 2)^{11}$

$t_8 = t_{7+1} = \binom{11}{7}(x^3)^4(-2)^7$

$= 330(x^{12})(-128)$

$= -42240x^{12}$

6. In the expansion of $\left(x^3 + \frac{2}{x}\right)^8$ find:

a) The number of terms

9

b) The general term

$t_{r+1} = \binom{8}{r}(x^3)^{8-r}\left(\frac{2}{x}\right)^r$

$= \binom{8}{r}(2^r)(x^{24-3r})(x^{-r})$

$= \binom{8}{r}(2^r)x^{24-4r}$

c) The third term

$t_3 = t_{2+1} = \binom{8}{2}(2)^2 x^{24-4(2)}$

$= 112x^{16}$

d) The constant term

$24-4r = 0$

$24 = 4r$

$6 = r$

$t_{6+1} = \binom{8}{6}(2^6)x^{24-4(6)}$

$= 1792x^0$

$= 1792$

7. In the expansion of $\left(x^3 - \frac{1}{x^2}\right)^{[redacted]\,10}$ find:

a) The number of terms

11

b) The general term

$t_{r+1} = \binom{10}{r}(x^3)^{10-r}\left(\frac{-1}{x^2}\right)^r$

$= \binom{10}{r}(-1)^r(x^{30-3r})(x^{-2r})$

$= \binom{10}{r}(-1)^r x^{30-5r}$

c) The third term

$t_3 = t_{2+1} = \binom{10}{2}(-1)^2 x^{30-5(2)}$

$= 45x^{20}$

d) The constant term

$30 - 5r = 0$

$30 = 5r$

$6 = r$

$t_{6+1} = \binom{10}{6}(-1)^6 x^{30-5(6)}$

$= 210x^0$

$= 210$

e) The term containing [redacted] x^5

$30 - 5r = 5$

$25 = 5r$

$5 = r$

$t_{5+1} = \binom{10}{5}(-1)^5 x^{30-5(5)}$

$= -252x^5$

8. Marshall is walking from his house to school. His route from home always takes him 4 blocks west and 9 blocks south to school but he likes to vary the path he takes.

a) How many different routes can he take?

$\binom{13}{4} = 715$

b) If Marshall needs to stop for a coffee at the Tim Horton's that is 2 blocks west and three blocks south of his house, how many routes pass by this store?

$\binom{5}{2}\binom{8}{2} = 10(28) = 280$

9. Consider a five-square game board. A checker is placed in one of the squares. Your piece is in the bottom row indicated by "x". Your checker is allowed to move one square at a time, diagonally left or right, to the row above. How many different paths will lead to the other side of the board? (You must do your work on this board.)

3		6		3
	3		3	
1		2		1
	1		1	
		X		

$= 3+6+3$

$= 12$ routes

10. What are the three main criteria that need to be satisfied in order to have a binomial distribution?

- Only 2 outcomes - PASS or FAIL
- all trials are identical
- trials are independent.

11. A basketball player has a shooting percentage of 0.450

a) Create a probability distribution table for the number of baskets made in a quarter where he takes 4 shots.

Number of Baskets Made (X)	P(X)
0	$\binom{4}{0}(0.45)^0(0.55)^4 = 0.0915$
1	$\binom{4}{1}(0.45)^1(0.55)^3 = 0.2995$
2	$\binom{4}{2}(0.45)^2(0.55)^2 = 0.3675$
3	$\binom{4}{3}(0.45)^3(0.55)^1 = 0.2005$
4	$\binom{4}{4}(0.45)^4(0.55)^0 = 0.0410$

b) What is the expected number of baskets made in the quarter?

$$E(X) = np$$
$$= 4(0.45)$$
$$= 1.8$$

12. The Choco-Latie Candies company makes candy-coated chocolates, 40% of which are red. The production line mixes the candies randomly and packages ten per box.

a) Calculate the probability that exactly 5 of the candies in a box are red.

$$P(X=5) = \binom{10}{5}(0.4)^5(0.6)^5$$
$$= 0.2007$$

b) Calculate the probability that at least 3 of the candies in a box are red.

$$P(X \geq 3) = 1 - [P(0) + P(1) + P(2)]$$
$$= 1 - \left[\binom{10}{0}(0.4)^0(0.6)^{10} + \binom{10}{1}(0.4)^1(0.6)^9 + \binom{10}{2}(0.4)^2(0.6)^8\right]$$
$$= 1 - 0.1673$$
$$= 0.8327$$

0.015

13. A certain type of rocket has a failure rate of 1.5%

a) Calculate the probability of there being exactly 1 failure in 100 launches. (answer to 6 decimal places)

$$= \binom{100}{1}(0.015)^1(0.985)^{99}$$

$$= 0.335953$$

b) Calculate the probability that there are more than 4 failures in 100 launches (answer to 6 decimal places)

$$P(X>4) = 1 - P(X \leq 4)$$

$$= 1 - \left[\binom{100}{0}(0.015)^0(0.985)^{100} + \binom{100}{1}(0.015)^1(0.985)^{99} + \binom{100}{2}(0.015)^2(0.985)^{98} + \binom{100}{3}(0.015)^3(0.985)^{97} + \binom{100}{4}(0.015)^4(0.985)^{96}\right]$$

$$= 0.017693$$

c) What is the expected number of failures in 100 launches of the rocket?

$$E(X) = np$$
$$= 100(0.015)$$
$$= 1.5$$

14. Suppose that 65% of the families in a town own computers. If eight families are surveyed at random...

a) What is the probability exactly 3 own a computer?

$$P(X=3) = \binom{8}{3}(0.65)^3(0.35)^5$$
$$= 0.0808$$

b) What is the probability that all 8 own a computer?

$$P(X=8) = \binom{8}{8}(0.65)^8(0.35)^0$$

$$= 0.0319$$

c) What is the probability that fewer than 6 families own a computer

$P(X<6) = 1 - P(X \geq 6)$

$= 1 - \left[\binom{8}{6}(0.65)^6(0.35)^2 + \binom{8}{7}(0.65)^7(0.35)^1 + \binom{8}{8}(0.65)^8(0.35)^0\right]$

$= 0.5722$

d) What is the expected number of families that will own a computer.

$E(X) = np$

$= 8(0.65)$

$= 5.2$

15. Suppose heads occurs 15 times in 20 tosses of a coin. Do you think the coin is fair? Explain your reasoning.

$P(X=15) = \binom{20}{15}(0.5)^{15}(0.5)^5$

$= 0.0148$

$= 1.48\%$

$E(X) = np$

$= 20(0.5)$

$= 10$

Expected number of heads is 10. Only 1.48% chance of having 15.

16. It is estimated that 15% of the Canadian population is undecided as to which political party to vote for in the next election. If a poll is conducted and 1007 citizens respond to the questionnaire, what is the probability that more than 100 of them are undecided? (Hint: Use the normal approximation of a binomial distribution)

Check

$np = 1007(0.15) = 151.05$ ✓

$n(1-p) = 1007(0.85) = 855.95$

$\bar{x} = 151.05$

$\sigma = \sqrt{1007(0.15)(0.85)}$

$= 11.33$

discrete ⟶ continuous

$P(X>100) = P(X>100.5)$

$= \text{normalCDF}(100.5, E99, 151.05, 11.33)$

$= 0.999996$

17. A bank found that 24% of its loans to new small businesses become delinquent. If 200 small businesses are selected randomly from the bank's files, what is the probability that at least 60 of them are delinquent? Compare the results from the normal approximation with the results from the calculations using a binomial distribution.

Check

$np = 200(0.24) = 48$ ✓

$n(1-p) = 200(0.76) = 152$ ✓

$\bar{x} = 48$

$\sigma = \sqrt{200(0.24)(0.76)}$

$= 6.04$

discrete $\rightarrow$ continuous

$P(X \geq 60) = P(X > 59.5)$

$= \text{normalCDF}(59.5, E99, 48, 6.04)$

$= 0.0285$

18. A recent survey of a gas-station's customers showed that 68% paid with credit cards, 29% used debit cards, and only 3% paid with cash. During her eight-hour shift as cashier at this gas station, Serena had a total of 223 customer.

a) What is the probability that...

i) at least 142 customers used a credit card?

Check

$np = 223(0.68) = 151.64$

$n(1-p) = 223(0.32) = 71.36$

$\bar{x} = 151.64$

$\sigma = \sqrt{223(0.68)(0.32)}$

$= 6.97$

discrete $\rightarrow$ continuous

$P(X \geq 142) = P(X > 141.5)$

$= \text{normalCDF}(141.5, E99, 151.64, 6.97)$

$= 0.9271$

ii) fewer than 220 customers paid with credit or debit cards

$0.68 + 0.29 = 0.97$

Check

$np = 223(0.97) = 216.31$ ✓

$n(1-p) = 223(0.03) = 6.69$ ✓

$\bar{x} = 216.31$

$\sigma = \sqrt{223(0.97)(0.03)}$

$= 2.55$

discrete $\rightarrow$ continuous

$P(X < 220) = P(X < 219.5)$

$= \text{normalCDF}(-E99, 219.5, 216.31, 2.55)$

$= 0.8945$

b) what is the expected number that paid Serena with cash?

$$E(X) = np$$
$$= 223(0.03)$$
$$= 6.7$$