Part 1: Perimeter and Area Relationships of a Rectangle

Optimization: the process of finding values that make a given quantity the greatest (or least) possible given certain conditions.

Investigation 1:

In this investigation you will determine how to find the maximum area of a rectangle with a fixed perimeter.

Remember: $Perimeter\ of\ a\ Rectangle = 2(l + w)$ **or** $2l + 2w$

$$Area\ of\ a\ Rectangle = length \times width$$

Instructions: Brandon needs to find the dimensions that will maximize the rectangular area of an enclosure with a perimeter of 32 meters.

a) Copy and complete the following table to determine the dimensions of the rectangle that maximizes the area of a rectangle that has a perimeter of 32 meters.

Rectangle	Width	Length	Perimeter	Area
1	1	15	32	15
2	2	14	32	28
3	3	13	32	39
4	4	12	32	48
5	5	11	32	55
6	6	10	32	60
7	7	9	32	63
8	8	8	32	64

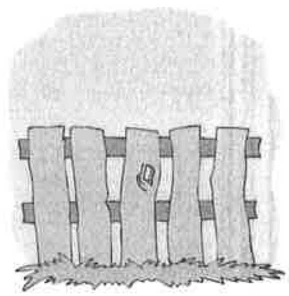
b) What are the dimensions of the rectangle with the maximum area? 8 x 8

c) What is the maximum area? 64 m^2

d) Describe the shape of the rectangle:

It is a square.

Problem 1



Your neighbour has asked for your advice about building his garden. He wants to fence the largest rectangular garden possible with 12 metres of fencing. What are the dimensions of the garden with the maximum area?

How can you organize the possible shapes so you don't miss one that might give the maximum area?

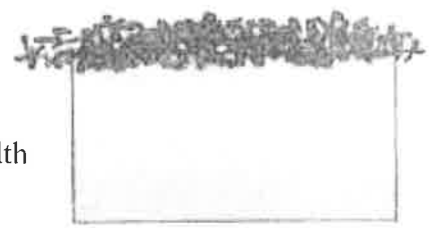
Length	Width	Perimeter	Area
1	5	12	5
2	4	12	8
* 3	3	12	9 *
4	2	12	8

Answer: The dimensions that give the maximum area of 3x3.
This gives an area of 9 m^2.

Investigation 2: In this investigation you will determine how to find the maximum area of a rectangle with a fixed sum of the lengths of three sides.

Instructions: Brandon's customer decides to use an existing hedge as one of the boundaries for the enclosure. This means that he will only use the prefabricated fencing on three sides of the rectangular enclosure. The client still wants the enclosure to have the greatest area possible. Brandon has 32 meters of prefabricated fencing.

width



length

a) Do you think Brandon will be able to enclose more, less, or the same amount of area now that the hedge is being used on one side? More.

b) What shape do you think will have the maximum area?

A rectangle with the length twice the width.

c) Copy and complete the chart to determine the dimensions of the rectangle that has the maximum area.

Rectangle	Width	Length	Sum of lengths of three sides	Area
1	1	30	32	30
2	2	28	32	56
3	3	26	32	78
4	4	24	32	96
5	5	22	32	110
6	6	20	32	120
7	7	18	32	126
8	8	16	32	128
9	9	14	32	126
10	10	12	32	120

d) What are the dimensions of the rectangle with maximum/optimal area? 8X16

e) What is the maximum area? $128\ m^2$

f) Describe the shape of the rectangle: The length is double the width.

Problem 2

A farmer is adding a rectangular corral to the side of a barn. The barn will form one side of the rectangle. The farmer has 16 meters of fencing to use. Conduct an investigation to determine the dimensions of the corral with maximum area.

$A = (l)(w)$

$P = l + 2w$

Width	Length	Perim.	Area
1	14	16	14
2	12	16	24
3	10	16	30
* 4	8	16	32 *

Width	Length	Perim	Area
5	6	16	30
6	4	16	24
7	2	16	14

The dimensions are: The dimensions that give max area are 4X8. The max area is $32m^2$.

Consolidation:

The dimensions of a rectangle that has maximum area depend on the number of sides to be fenced.

When fencing all four sides of a rectangle, the optimal shape is a square.

When only three sides are being fenced, the relationship between the length and the width of the rectangle is l = 2w.

Apply What You Learned

1) What dimensions will provide the maximum area for a rectangle with each perimeter?

a) 20 m $20 \div 4 = 5$

$\boxed{\text{A 5m} \times \text{5m square}}$

b) 36 m

$36 \div 4 = 9$

$\boxed{\text{A 9m} \times \text{9m square}}$

c) 50 m

$50 \div 4 = 12.5$

$\boxed{\text{A 12.5m} \times \text{12.5m square}}$

d) 83 m

$83 \div 4 = 20.75$

$\boxed{\text{A 20.75m} \times \text{20.75m square}}$

2. Cam is building a fenced-in play yard for his small children against one wall of his house. He has 12 m of fencing. What whole-number dimensions produce the play yard with the greatest area?

using a chart:

width	Length	Perimeter	Area
1	10	12	10
2	8	12	16
* 3	6	12	18 *
4	4	12	16
5	2	12	10

Note: $P = l + 2w$

$A = (l)(w)$

using algebra:

$P = l + 2w$ but we know $l = 2w$

so $P = 2w + 2w$ we know $P = 12$

$12 = 4w$

$\frac{12}{4} = w$

$3 = w$

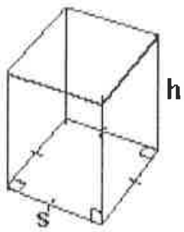
If $w=3$ and we know area is maximized when the $l=2w$; the length must be 6.

The dimensions that maximize the area are 3m by 6m.

The max area is $18m^2$.

Part 2: Minimize the Surface Area of a Square Based Prism

Recall:



$$Volume\ of\ a\ Prism = (Area\ of\ Base)(Height)$$

$$Volume\ of\ Square\ Based\ Prism = (s^2)(h)$$

$$Surface\ Area\ of\ a\ Square\ Based\ Prism = 2s^2 + 4sh$$

Minimizing surface area for a given volume is important when designing packages and containers to save on materials and reduce heat loss.

Investigation: You will determine how to find the minimum surface area of a square based prism for a given volume.

a) Complete the following table to determine the dimensions of the square based prism that has a minimum surface are for a fixed volume of 64 m^3

Note: You can calculate height given the volume and side length using $h = \frac{volume}{length^2}$

	Side Length of Square Base (s)	Area of Square base (s^2)	Height (h)	Volume (s^2h)	Surface Area ($2s^2 + 4sh$)
1	1	1	64	64	258
2	2	4	16	64	136
3	3	9	7.1	64	103.2
4	4	16	4	64	96
5	5	25	2.56	64	101.2
6	6	36	1.78	64	114.72
7	7	49	1.3	64	134.4

b) Which prism has the least surface area?

Prism # 4.

c) Describe the shape of this prism

The prism is a cube. The side length of the square base = the height.

s=h.

Results:

For a square based prism with a given volume, the minimum surface area occurs when the prism is a cube.

Given a volume, you can find the dimensions of a square-based prism with minimum surface area by solving for s in the formula $V = s^3$, This is the formulas for the volume of a cube where V is the given volume and s is the length of a side of the cube. You can solve for the side length by taking the cubed root of the volume.

For example: The cubed root of 64 = $\sqrt[3]{64} = 64^{1/3} = 4$

Apply What You Learned

1) a) What would be the dimensions of the square-based prism with a minimum surface area given a volume of 27 cubic units?

$V = s^3$

$27 = s^3$

$\sqrt[3]{27} = s$

$3 = s$

The dimensions would be 3 x 3 x 3 .

b) Calculate the surface area ($Surface\ area\ of\ a\ cube = 6s^2$)

$SA = 6(3)^2$

$= 6(9)$

$= 54$

If a square based prism has a volume of 27units^3; the minimum surface area is 54 units^2.

2) Talia is shipping USB cables in a small cardboard square-based prism box. The box must have a capacity of 750cm^3 and Talia wants to use the minimum amount of cardboard when she ships the box.

a) What should the dimensions of the box be, to the nearest hundredth of a centimeter?

$V_{cube} = s^3$

$750 = s^3$

$\sqrt[3]{750} = s$

$9.09 = s$

The dimensions would be 9.09cm x 9.09cm x 9.09cm.

b) What is the minimum amount of cardboard that Talia will need, to the nearest tenth of a square centimeter? ($Surface\ area\ of\ a\ cube = 6s^2$)

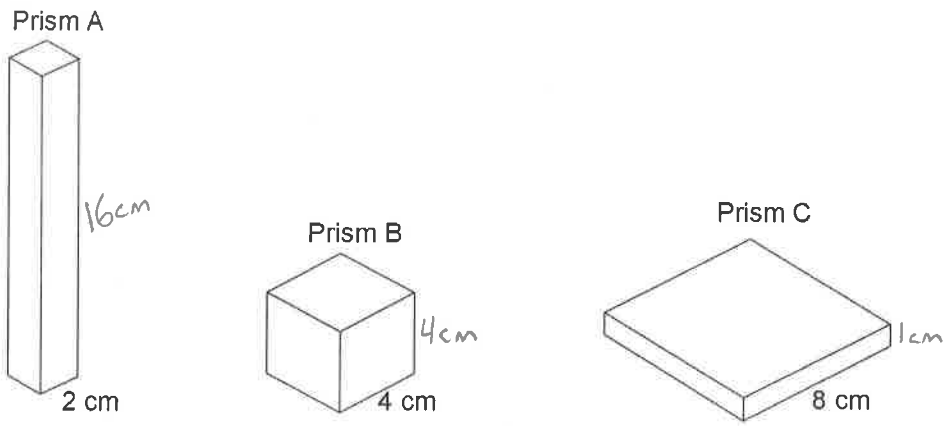
$SA = 6s^2$

$SA = 6(9.09)^2$

$SA = 495.8$

The minimum amount of cardboard needed is 495.8 cm^2.

3) Each of these square-based prisms has volume 64 cm^3.



a) Find the height of each prism.

$\boxed{h = \frac{\text{volume}}{s^2}}$

Prism A:

$h = \frac{64}{(2)^2}$

$= 16$ cm

Prism B:

$h = \frac{64}{(4)^2}$

$= 4$ cm

Prism C:

$h = \frac{64}{(8)^2}$

$= 1$ cm

b) Find the surface area of each prism.

$\boxed{SA = 2s^2 + 4sh}$

Prism A:

$SA = 2(2)^2 + 4(2)(16)$

$= 8 + 128$

$= 136 \text{cm}^2$

Prism B:

$SA = 2(4)^2 + 4(4)(4)$

$= 32 + 64$

$= 96 \text{cm}^2$

Prism C:

$SA = 2(8)^2 + 4(8)(1)$

$= 128 + 32$

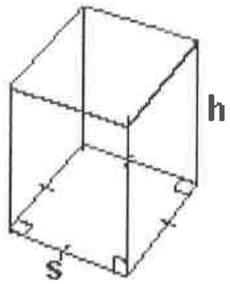
$= 160 \text{cm}^2$

c) Order the prisms from least to greatest surface area.

B, A, C

Part 3: Maximize the Volume of a Square-Based Prism

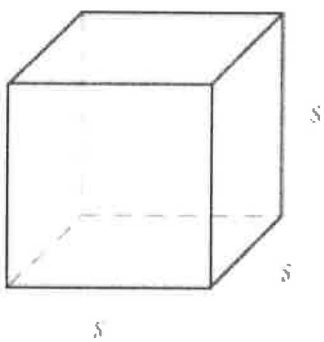
Square-based prism



$Volume = s^2h$

$Surface\ Area = 2s^2 + 4sh$

Cube



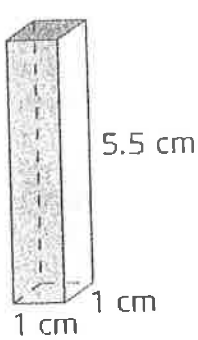
$Volume = s^3$

$Surface\ Area = 6s^2$

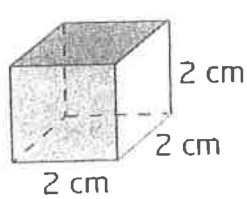
For a given surface area, it is usually optimal to maximize. the volume.

Investigation 1: Each of the square-based prisms has a surface area of 24 cm^2.

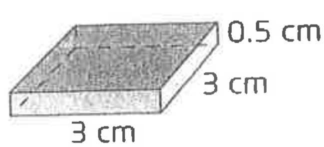
Prism 1



Prism 2



Prism 3



a) Calculate the area of the base and the volume of each prism. Record your data in the following table:

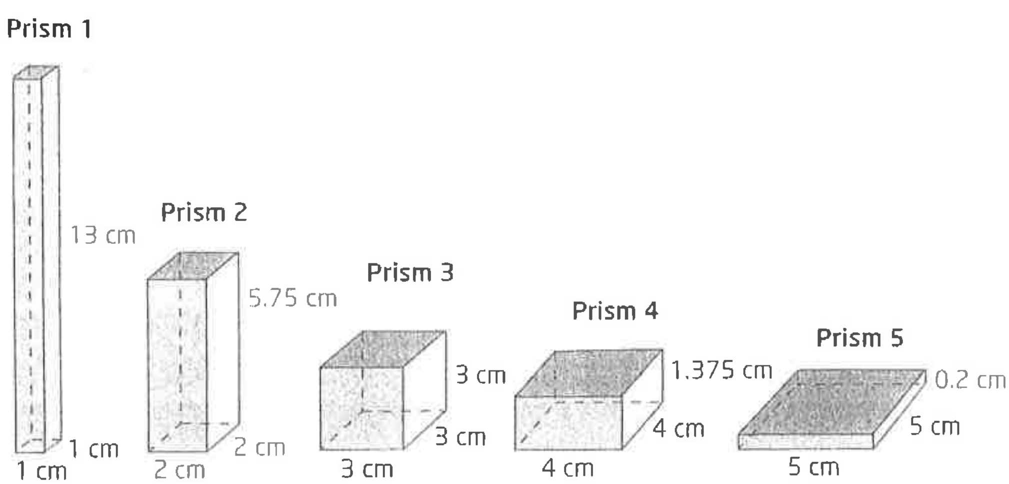
Prism Number	Length of base	Area of base	Surface Area	Height	Volume
1	1	1	24	5.5	5.5
2	2	4	24	2	8
3	3	9	24	0.5	4.5

b) Which square-based prism has the maximum volume? Describe the shape of this prism compared to the others.

Prism 2 has max volume for the given surface area.

The prism is a cube. s=h.

Investigation 2: Each of the square-based prisms has a surface area of $54cm^2$.



a) Predict the dimensions of the square-based prism with maximum volume if the surface area is $54cm^2$.

Prism 3 will have the max volume for the given surface area because it is a cube.

b) Test your prediction by completing the following table.

Prism Number	Length of base	Area of base	Surface Area	Height	Volume
1	1	1	54	13	13
2	2	4	54	5.75	23
3	3	9	54	3	27
4	4	16	54	1.375	22
5	5	25	54	0.2	5

Results: The maximum volume for a given surface area of a square-based prism always occurs when the prism is a cube.

Example 1: Determine the dimensions of the square-based prism with maximum volume that can be formed using 5400 cm^2 of cardboard.

Solution:

We know that a cube will have the maximum volume

$Surface\ Area\ of\ a\ Cube = 6s^2$ where s represents the side length

To calculate the side length of the cube given the surface area you must rearrange the formula to isolate s:

$$s = \sqrt{\frac{Surface\ Area}{6}}$$

Therefore, if the Surface Area is 5400 cm^2, $s = \sqrt{\frac{5400}{6}} = 30cm$. This means that the dimensions of the square-based prism with maximum volume are **30cm x 30cm x 30cm.**

Apply What You Learned

1. Determine the dimensions of the square-based prism with maximum volume for each surface area.

a) 150 cm^2

$s = \sqrt{\frac{SA}{6}}$

$= \sqrt{\frac{150}{6}}$

$= 5$

$\boxed{\text{The dimensions are 5cm} \times \text{5cm} \times \text{5cm}}$

b) 2400 m^2

$s = \sqrt{\frac{SA}{6}}$

$= \sqrt{\frac{2400}{6}}$

$= 20$

$\boxed{\text{The dimensions are 20m} \times \text{20m} \times \text{20m}}$

c) 750 cm^2

$s = \sqrt{\frac{SA}{6}}$

$= \sqrt{\frac{750}{6}}$

$= 11.2$

$\boxed{\text{The dimensions are 11.2cm} \times \text{11.2cm} \times \text{11.2cm}}$

2. Determine the volume of each prism in question 1, to the nearest cubic unit.

a) $V = s^3$

$= (5)^3$

$= 125 \text{ cm}^3$

b) $V = s^3$

$= (20)^3$

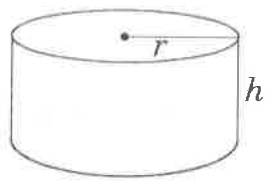
$= 8000 \text{ m}^3$

c) $V = s^3$

$= (11.2)^3$

$= 1404.9 \text{ cm}^3$

Part 4: Maximize the Volume of a Cylinder

Cylinder 	$A_{base} = \pi r^2$ $A_{lateral\ surface} = 2\pi rh$ $A_{total} = 2A_{base} + A_{lateral\ surface}$ $= 2\pi r^2 + 2\pi rh$	$V = (A_{base})(\text{height})$ $V = \pi r^2 h$

Investigation 1: Maximize Volume of a Cylinder for a Given Surface Area

Your task is to design a cylindrical juice can that uses 470cm² of aluminum. The can should have the greatest volume possible.

a) Substitute the given surface area and radius into the surface area formula and solve for the height, h. You can use the rearranged formula [redacted] $h = \frac{SA - 2\pi r^2}{2\pi r}$

b) Determine the volume of this can using the formula for the volume of a cylinder: $V = \pi r^2 h$. Record the data in the following table:

Surface Area	Radius	Height	Volume
470	2	35.4	444.8
470	5	10.0	785.4
470	8	1.4	281.5

c) What is the maximum volume for the cans in your table? What are the radius and height for this can?

The max volume is 785.4 cm^3.

radius = 5cm
height = 10cm

Conclusion: The maximum volume for a given surface area of a cylinder occurs when its height equals its diameter. That is, $h =$ d or $h = 2$r

Example 1:

a) Determine the dimensions of the cylinder with maximum volume that can be made with $600cm^2$ of aluminum. Round the dimensions to the nearest hundredth of a centimeter.

Solution:

We know that a cylinder with height equal to the diameter (double the radius) gives the maximum volume for a given surface area.

To determine the radius of the can with a given surface area, use the surface area formula and substitute h = 2r.

Given: $SA = 2\pi r^2 + 2\pi rh$ and $h = diameter = 2r$

$$SA = 2\pi r^2 + 2\pi r(2r)$$
$$SA = 2\pi r^2 + 4\pi r^2$$
$$SA = 6\pi r^2$$

To solve for the radius given a surface area, you must use the rearranged formula: $r = \sqrt{\frac{SA}{6\pi}}$

$$r = \sqrt{\frac{600}{6\pi}} = 5.64$$

The radius of the cylinder should be 5.64 cm and the height should be twice that, or 11.28 cm.

b) Determine the volume of this cylinder

$$Volume = \pi r^2 h$$
$$Volume = \pi(5.64)^2(11.28)$$
$$Volume = 1127\ cm^3$$

Apply What You Learned

1. Determine the dimensions of the cylinder with the maximum volume for each surface area. Round the dimensions to the nearest hundredth of a unit.

a) 1200 cm^2

$r = \sqrt{\frac{SA}{6\pi}}$

$r = \sqrt{\frac{1200}{6\pi}}$

$r = 7.98cm$

$h = d$
$= 2r$
$= 2(7.98)$
$= 15.96cm$

radius = 7.98cm
height = 15.96cm

b) 125 cm^2

$r = \sqrt{\frac{SA}{6\pi}}$

$r = \sqrt{\frac{125}{6\pi}}$

$r = 2.58cm$

$h = d$
$= 2r$
$= 2(2.58)$
$= 5.16cm$

radius = 2.58cm
height = 5.16cm.

2. Determine the volume of each cylinder in question 1. Round to the nearest whole unit.

a) $V = \pi r^2 h$
$= \pi(7.98)^2(15.96)$
$= 3192.92\ cm^3$

$\boxed{3193\ cm^3}$

b) $V = \pi r^2 h$
$= \pi(2.58)^2(5.16)$
$= 107.9\ cm^3$

$\boxed{108\ cm^3}$