L1 – Equations of Lines in Two-Space Unit 6
MCV4U
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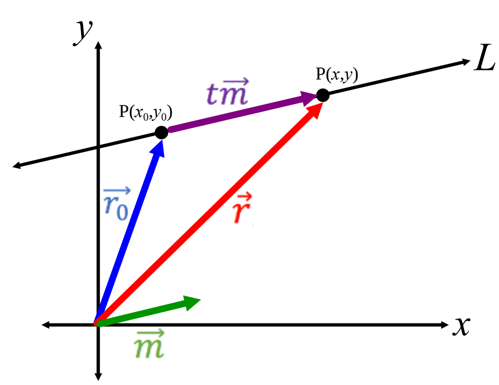
In two-space, a line can be defined many different types of equations:

i)	Slope y-intercept form	→	$y = mx + b$
ii)	Standard form (scalar equation)	→	$Ax + By + C = 0$
iii)	Vector Equation	→	$\vec{r} = \vec{r_0} + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$
iv)	Parametric Equation	→	$\begin{cases} x = x_0 + tm_1 \\ y = y_0 + tm_2 \end{cases}$

Part 1: Vector Equation of a Line in Two-Space

In slope y-intercept form and standard form, we have equations that define all points (x, y) on the line. A vector equation of a line is an equation that describes resultant vectors that start at the origin and end at a point on the line. In order to create these resultant vectors, we need a position vector that gives us a point on the line, $\vec{r_0} = [x_0, y_0]$, and a direction vector parallel to the line, $\vec{m} = [m_1, m_2]$. By adding the position and direction vectors together, we get a resultant vector, $\vec{r}$, that has its tip on a point on the line. By multiplying the direction vector by t (to get $t\vec{m}$), the resultant vector, $\vec{r}$, can have its tip at any point on the line.



Summary of Vector Equation of a Line:

$\vec{r} = \vec{r_0} + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$

where

- $t \in \mathbb{R}$
- $\vec{r} = [x, y]$ is a position vector to any unknown point on the line
- $\vec{r_0} = [x_0, y_0]$ is a position vector to any known point on the line
- $\vec{m} = [m_1, m_2]$ is a direction vector parallel to the line

Example 1: For a line that goes through the points $A(1,4)$ and $B(3,1)$

a) Write a vector equation for the line

Find direction vector $\vec{m}$:

$\vec{m} = \overrightarrow{AB}$

$\vec{m} = [3-1, 1-4]$

$\vec{m} = [2, -3]$

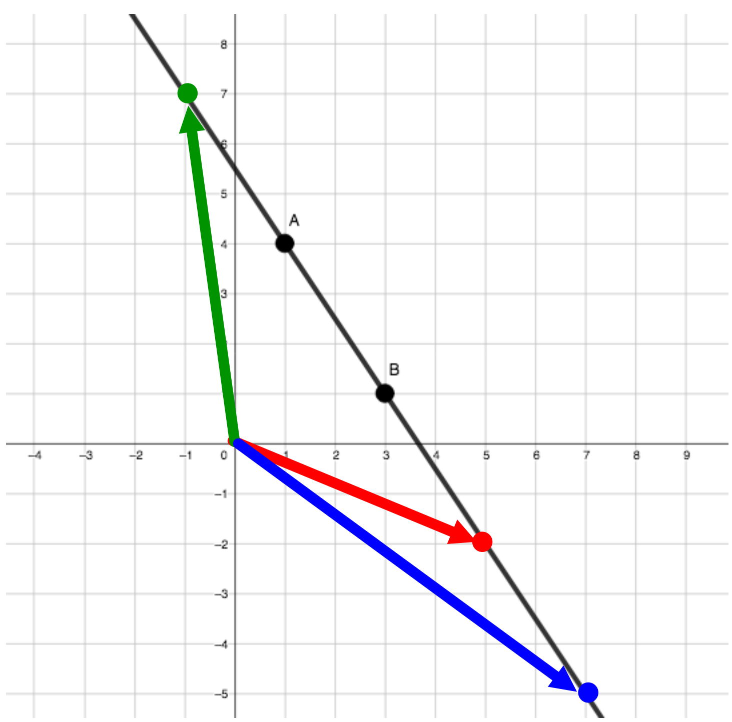
Use either point $A(1,4)$ or $B(3,1)$ for $\vec{r_0}$

A vector equation is: $[x, y] = [3,1] + t[2, -3]$

b) Determine three more position vectors to points on the line. Graph the line.

https://www.geogebra.org/graphing/j6ntqajv

$t = 1$	$t = 2$	$t = -2$
$[x, y] = [3,1] + 1[2, -3]$	$[x, y] = [3,1] + 2[2, -3]$	$[x, y] = [3,1] + (-2)[2, -3]$
$[x, y] = [5, -2]$	$[x, y] = [7, -5]$	$[x, y] = [-1,7]$



c) Determine if the point (2,3) is on the line.

If (2,3) is on the line, there is a value of t that makes the following equation true:

$[2,3] = [3,1] + t[2,-3]$

Equate the x-coordinates:	Equate the y-coordinates:
$2 = 3 + 2t$	$3 = 1 - 3t$
$t = -\frac{1}{2}$	$t = -\frac{2}{3}$

Since the t-values are not equal, the point (2,3) is NOT on the line.

Part 2: Parametric Equation of a Line in Two-Space

Remember that the Vector Equation of a line can be written as:

i) $\vec{r} = \overrightarrow{r_0} + t\vec{m}$ or $[x, y] = [x_0, y_0] + t[m_1, m_2]$

The vector equation of the line can be separated in to two parts, one for each variable.

The parametric equations of a line in two-space are:

$$x = x_0 + tm_1$$
$$y = y_0 + tm_2$$

The parametric equation of the line from example 1 would be:

$$\ell: \begin{cases} x = 3 + 2t \\ y = 1 - 3t \end{cases}$$

Example 2: Consider the line $\ell_1: \begin{cases} x = 3 + 2t \\ y = -5 + 4t \end{cases}$

a) Find the coordinates of two points on the line.

For Point 1, let $t = 0$:

$x = 3 + 2(0)$ $\quad$ $y = -5 + 4(0)$

$x = 3$ $\quad$ $y = -5$

$P_1 = (3, -5)$

For Point 2, let $t = 1$:

$x = 3 + 2(1)$ $\quad$ $y = -5 + 4(1)$

$x = 5$ $\quad$ $y = -1$

$P_2 = (5, -1)$

b) Write a vector equation of the line.

$\vec{r_0} = [3, -5]$

$\vec{m} = [2,4] = 2[1,2]$ (use reduced version of direction vector if possible)

The vector equation is: $[x, y] = [3, -5] + t[1,2]$

c) Write the scalar equation of the line.

To write the scalar equation, isolate t in both parametric equations, then set them equal to each other and rearrange in to standard form.

$x = 3 + t$ $\quad$ $y = -5 + 2t$

$t = x - 3$ $\quad$ $t = \dfrac{y+5}{2}$

$x - 3 = \dfrac{y+5}{2}$

$2x - 6 = y + 5$

$2x - y - 11 = 0$

d) Determine if ℓ_1 is parallel to ℓ_2: $\begin{cases} x = 1 + 3t \\ y = 8 + 12t \end{cases}$

If they are parallel, the direction vectors will be scalar multiples of each other.

$[2,4] = k[3,12]$

$2 = 3k$

$k = \frac{2}{3}$

$4 = 12k$

$k = \frac{1}{3}$

Therefore, ℓ_1 and ℓ_2 are not parallel.

Example 3: Consider the line with scalar equation $4x + 5y + 20 = 0$

a) Graph the line

x-int:

$4x + 5(0) + 20 = 0$

$x = -5$

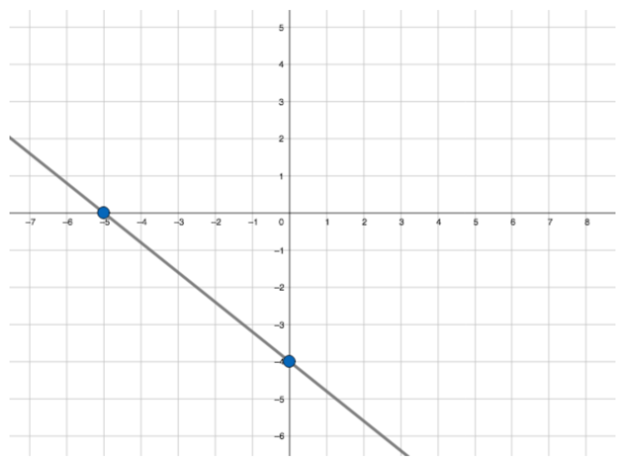
$(-5,0)$

y-int:

$4(0) + 5y + 20 = 0$

$y = -4$

$(0,-4)$



b) Determine a position vector that is perpendicular to the line.

Start by finding a direction vector for the line:

$\vec{m} = [0, -4] - [-5, 0]$

$\vec{m} = [5, -4]$

To find a vector $\vec{n}$ that is perpendicular to $\vec{m}$, set $\vec{m} \cdot \vec{n} = 0$ and solve for values of x and y that make it true:

$[5, -4] \cdot [x, y] = 0$

$5x - 4y = 0$

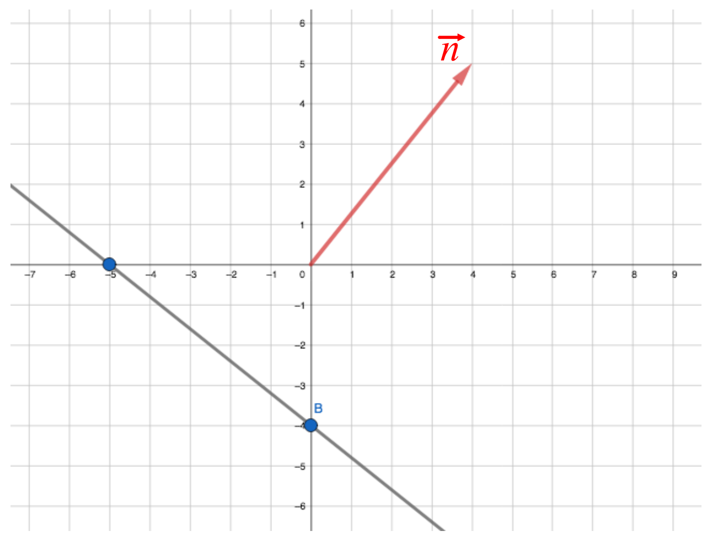
$x = \frac{4}{5}y$

Choose any value for y and then solve for x

$x = \frac{4}{5}(5)$

$x = 4$

$\vec{n} = [4, 5]$ is perpendicular to the line.



c) How does the position vector from part b) compare to the scalar equation?

The components of $\vec{n}$ correspond to the coefficients of x and y in the scalar equation.

d) Write a vector equation of the line

$\vec{r_0} = [0, -4]$

$\vec{m} = [5, -4]$

The vector equation is: $[x, y] = [0, -4] + t[5, -4]$

Example 4: Find a scalar equation for the line $[x, y] = [0,3] + t[2, -1]$

Perpendicular vector:

Remember shortcut for finding a perpendicular vector is to swap coordinates and change 1 sign.

$\vec{n} = [1,2]$

$x + 2y + C = 0$

$0 + 2(3) + C = 0$

$C = -6$

Scalar equation is $x + 2y - 6 = 0$