L2 – The Product Rule Unit 1
MCV4U
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Part 1: Proof of the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = \left[\lim_{h\to 0} f(x+h)\right]\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right] + \left[\lim_{h\to 0} g(x)\right]\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right]$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

The Product Rule:

If $P(x) = f(x)g(x)$, then $P'(x) = f'(x)g(x) + g'(x)f(x)$

"Derivative of the first times the second plus derivative of the second times the first"

Part 2: Apply the Product Rule

Example 1: Use the product rule to differentiate each function.

a) $P(x) = (3x-5)(x^2+1)$

$$P'(x) = 3(x^2+1) + 2x(3x-5)$$

$$P'(x) = 3x^2 + 3 + 6x^2 - 10x$$

$$P'(x) = 9x^2 - 10x + 3$$

b) $y = (2x+3)(1-x)$

$$\frac{dy}{dx} = 2(1-x) + (-1)(2x+3)$$

$$\frac{dy}{dx} = 2 - 2x - 2x - 3$$

$$\frac{dy}{dx} = -4x - 1$$

Example 2: Find $h'(-1)$ where $h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$

$h'(x) = (15x^2 + 14x)(2x^2 + x + 6) + (4x + 1)(5x^3 + 7x^2 + 3)$

$h'(-1) = [15(-1)^2 + 14(-1)][2(-1)^2 + (-1) + 6] + [4(-1) + 1][5(-1)^3 + 7(-1)^2 + 3]$

$h'(-1) = (1)(7) + (-3)(5)$

$h'(-1) = -8$

Example 3: Find the derivative of $g(x) = (x - 1)(2x)(x^2 + 3)$

$g'(x) = (4x - 2)(x^2 + 3) + (2x)(x - 1)(2x)$

$g'(x) = 4x^3 + 12x - 2x^2 - 6 + 4x^2(x - 1)$

$g'(x) = 4x^3 + 12x - 2x^2 - 6 + 4x^3 - 4x^2$

$g'(x) = 8x^3 - 6x^2 + 12x - 6$

Consider $(x - 1)(2x)$ as the 1st function

Consider $x^2 + 3$ as the 2nd function

$\frac{d}{dx}$ 1st $= 1(2x) + 2(x - 1) = 4x - 2$

Note: In example 3, expanding first would probably be easier, but that is not always the case such as with $h(x) = (2x) \cdot \sqrt{x + 1}$

Example 4: Determine an equation for the tangent to the curve $y = (x^2 - 1)(x^2 - 2x + 1)$ at $x = 2$.

Point on the tangent line:

$y = [2^2 - 1][2^2 - 2(2) + 1]$

$y = (3)(1)$

$y = 3$

$(2, 3)$

Slope of tangent line:

$\frac{dy}{dx} = 2x(x^2 - 2x + 1) + (2x - 2)(x^2 - 1)$

$\left.\frac{dy}{dx}\right|_{x=2} = 2(2)[2^2 - 2(2) + 1] + [2(2) - 2][2^2 - 1]$

$\left.\frac{dy}{dx}\right|_{x=2} = 4(1) + 2(3)$

$\left.\frac{dy}{dx}\right|_{x=2} = 10$

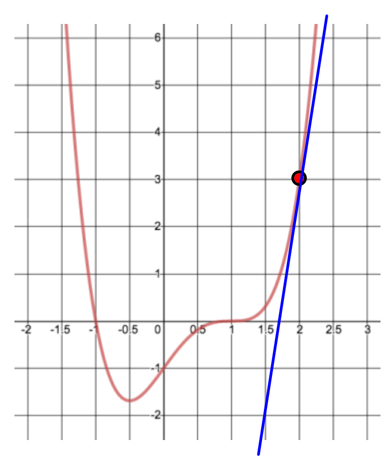
Equation of Tangent Line:

$y = mx + b$

$3 = 10(2) + b$

$b = -17$

$y = 10x - 17$



Example 5: Student council is organizing its annual trip to an out-of-town concert. For the past 3 years, the cost of the trip has been \$140 per person. At this price, all 200 seats on the train were filled. This year, student council plans to increase the price of the trip. Based on a student survey, council estimates that for every \$10 increase in price, five fewer students will attend the concert.

a) Write an equation to represent revenue, R, in dollars, as a function of the number of \$10 increases, n.

$R(n) = (price)(\#\ of\ students)$
$R(n) = (140 + 10n)(200 - 5n)$

b) Determine an expression, in simplified form, for $\frac{dR}{dn}$ and interpret it for this situation.

$R'(n) = 10(200 - 5n) + (-5)(140 + 10n)$

$R'(n) = 2000 - 50n - 700 - 50n$

$R'(n) = -100n + 1300$

c) Determine when $R'(n) = 0$. What information does this give the manager?

$0 = -100n + 1300$

$100n = 1300$

$n = 13$

The tangent slope is 0 when $n = 13$. Therefore, there is a maximum revenue when there are 13 price increases.

$R(13) = [140 + 10(13)][200 - 5(13)]$

$R(13) = (270)(135)$

$R(13) = 36450$

With 13 price increases, the manager will sell 135 tickets for \$270 each and make a max revenue of \$36450.