L5 – Intersections of Lines in 2-Space and 3-Space Unit 6
MCV4U
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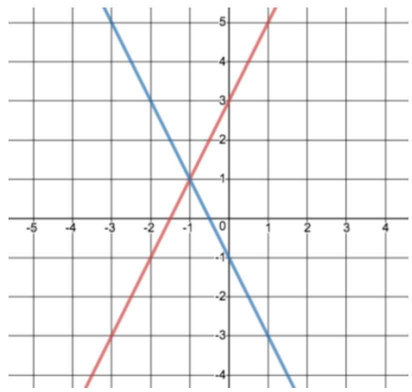
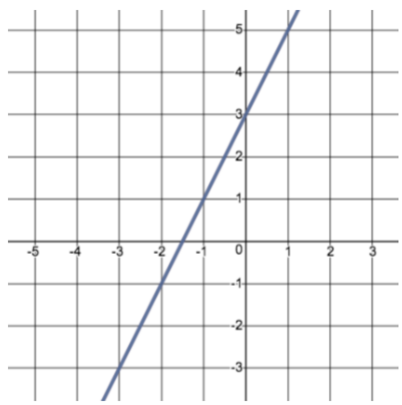
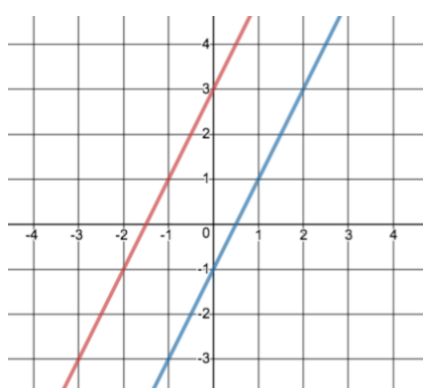
Part 1: Intersection of Lines in 2-Space

Possibilities for intersection of lines in 2-Space:

Intersections	Graph	Equations
1 intersection at a point		$y = 2x + 3$ $y = -2x - 1$
Infinite Points of Intersection (parallel and coincident)		$y = 2x + 3$ $y = 2x + 3$
No points of intersection (parallel and distinct)		$y = 2x + 3$ $y = 2x - 1$

Note: Linear systems in 2-space can be solved using substitution or elimination.

Example 1: Find the solutions of each system if they exist.

a) $\ell_1: 4x - 6y = -10$

$\ell_2: 6x - 9y = -15$

$3 \times ①$ $\quad 12x - 18y = -30$

$2 \times ②$ $\quad 12x - 18y = -30 \quad -$

$0 = 0$

Therefore, there are infinitely many solutions because these lines are parallel and coincident (same line).

b) $\ell_1: [x, y] = [1,5] + s[-6,8]$

$\ell_2: [x, y] = [2,1] + t[9, -12]$

Write in parametric form:

$\ell_1: \begin{cases} x = 1 - 6s \\ y = 5 + 8s \end{cases}$ $\qquad \ell_2: \begin{cases} x = 2 + 9t \\ y = 1 - 12t \end{cases}$

Equate the x and y variables:

① $1 - 6s = 2 + 9t$ $\qquad$ ② $5 + 8s = 1 - 12t$

① $-1 = 9t + 6s$ $\qquad$ ② $-4 = 12t + 8s$

Solve using elimination:

$4 \times ①$ $\quad -4 = 36t + 24s$

$3 \times ②$ $\quad -12 = 36t + 24s \quad -$

$8 = 0$

Therefore, there are no solutions to this system. The lines are parallel and distinct.

c) $\ell_1: [x, y] = [3,4] + s[3, -2]$

$\ell_2: [x, y] = [-2, -2] + t[3,5]$

Write in parametric form:

$\ell_1: \begin{cases} x = 3 + 3s \\ y = 4 - 2s \end{cases}$ $\ell_2: \begin{cases} x = -2 + 3t \\ y = -2 + 5t \end{cases}$

Equate the x and y variables:

① $3 + 3s = -2 + 3t$ ② $4 - 2s = -2 + 5t$

① $5 = 3t - 3s$ ② $6 = 5t + 2s$

Now solve using elimination:

$2 \times$① $10 = 6t - 6s$

$3 \times$② $18 = 15t + 6s$ $+$

$28 = 21t$

$t = \frac{4}{3}$

Sub $t = \frac{4}{3}$ in to ℓ_2:

$x = -2 + 3\left(\frac{4}{3}\right)$ $y = -2 + 5\left(\frac{4}{3}\right)$

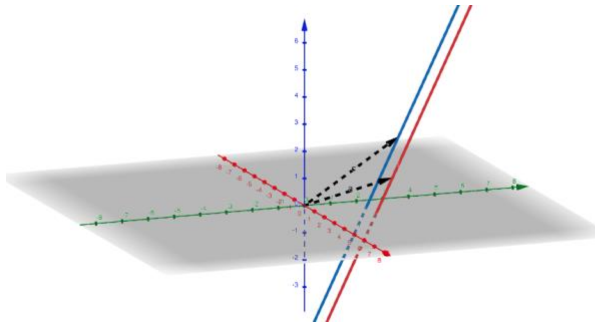
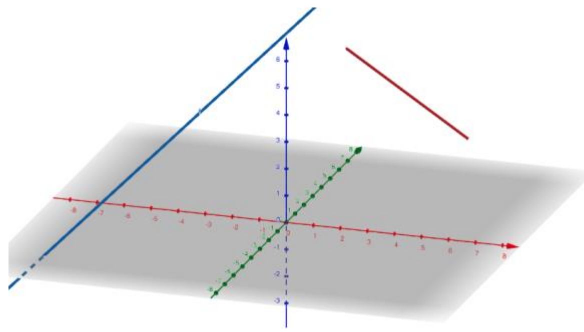
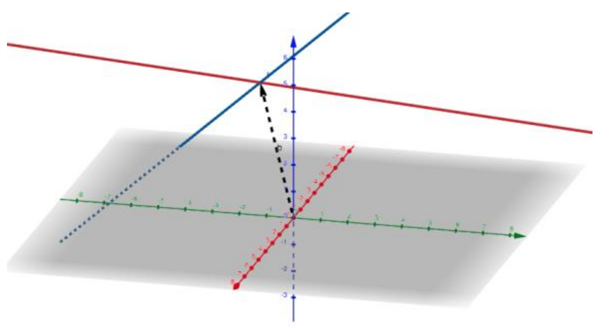
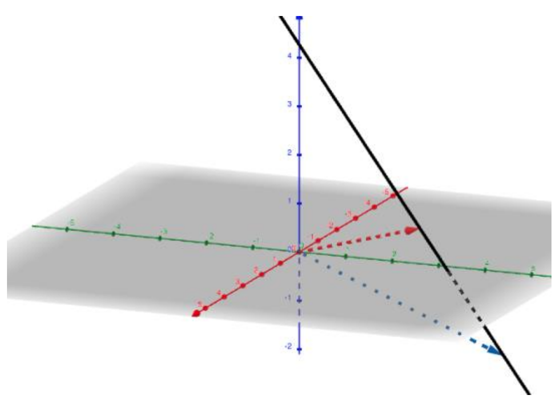
$x = 2$ $y = \frac{14}{3}$

There is one unique solution at $\left(2, \frac{14}{3}\right)$

Note: *You could at this point solve for s and make sure you get the same solution. This step will be necessary when solving systems in 3-space.*

Part 2: Intersection of Lines in 3-Space

Possibilities for intersection of lines in 3-Space:

Intersections	Graph	Equations
No intersections (parallel and distinct)		$\vec{a} = [-1,4,2] + s[-2,2,2]$ $\vec{b} = [1,3,1] + t[-2,2,2]$
No intersections (not parallel but skewed)		$\vec{a} = [-3,-2,4] + s[1,1,1]$ $\vec{b} = [2,5,4] + t[-2,4,-1]$
1 intersection (not parallel but on same plane)		$\vec{a} = [-3,-2,4] + s[1,1,1]$ $\vec{b} = [-3,-2,4] + t[-2,4,-1]$
Infinite points of intersection (parallel and coincident)		$\vec{a} = [-1,4,-2] + s[-2,1,-3]$ $\vec{b} = [1,3,1] + t[-2,1,-3]$

Steps for solving the intersection of lines in 3-space:

The first step is always to check if direction vectors are parallel or not by comparing direction vectors and seeing if one is a scalar multiple of the other ($\overrightarrow{m_1} = k\overrightarrow{m_2}$)

a. If **yes**, there could be infinite solutions OR no solutions. Test to see if a point is on both lines or just solve the system and see if you get no solutions or infinite solutions.

b. If **no**, they could intersect at a point OR they could be skewed and never intersect. Try solving the system and see if you can find parameters that result in the equations being equal. If you can, plug either parameter in to an original line and solve for the point of intersection.

Example 2: Determine if the lines intersect. If they do, find the coordinates of the point of intersection.

a) $\ell_1: [x, y, z] = [5,11,2] + s[1,5,-2]$

$\ell_2: [x, y, z] = [1,-9,9] + t[2,10,-4]$

Step 1: Check if lines are parallel.

$[1,5,-2] = k[2,10,-4]$

$k = \frac{1}{2}$ satisfies this equation, therefore they are parallel.

Step 2: Test to see if $(5,11,2)$ is on ℓ_2. If it is on both lines, then the lines are coincident (infinite solutions). If it is not on both lines, the lines are parallel and distinct (no solutions)

$$\ell_2: \begin{cases} x = 1 + 2t \\ y = -9 + 10t \\ z = 9 - 4t \end{cases}$$

$5 = 1 + 2t$ $\qquad$ $11 = -9 + 10t$ $\qquad$ $2 = 9 - 4t$

$t = 2$ $\qquad$ $t = 2$ $\qquad$ $t = \frac{7}{4}$

Since there is not a single value of t that satisfies this equation, point (5,11,2) is not on ℓ_2 and therefore the lines are parallel and distinct. This means there are no solutions.

This was a geometric reasoning solution. You could solve this algebraically as well.

b) $\ell_1: [x, y, z] = [7,2,-6] + s[2,1,-3]$

$\ell_2: [x, y, z] = [3,9,13] + t[1,5,5]$

$[2,1,-3] \neq k[1,5,5]$; therefore, these lines are not parallel and either intersect once or not at all (skew).

Write in parametric form:

$$\ell_1: \begin{cases} x = 7 + 2s \\ y = 2 + s \\ z = -6 - 3s \end{cases} \qquad \ell_2: \begin{cases} x = 3 + t \\ y = 9 + 5t \\ z = 13 + 5t \end{cases}$$

Equate the variables:

① $7 + 2s = 3 + t$ ② $2 + s = 9 + 5t$ ③ $-6 - 3s = 13 + 5t$

① $4 = t - 2s$ ② $-7 = 5t - s$ ③ $-19 = 5t + 3s$

Solve for t and s using elimination with any pair of two equations:

① $4 = t - 2s$

2 ×② $-14 = 10t - 2s$ −

$18 = -9t$

$t = -2$

Sub $t = -2$ in to ① or ②

$-7 = 5(-2) - s$

$s = -3$

Now you MUST check to see if these values of s and t satisfy equation ③ as well:

LS

$= -19$

RS

$= 5(-2) + 3(-3)$

$= -19$

LS = RS. Therefore, these lines intersect at a single point. Sub $s = -3$ in to ℓ_1 or $t = -2$ in to ℓ_2 to find the point:

$\ell_1: [x, y, z] = [7,2,-6] - 3[2,1,-3]$

$[x, y, z] = [1,-1,3]$

The point of intersection is $(1, -1, 3)$

https://www.geogebra.org/3d/esvcf8kf

c) $\ell_1: [x, y, z] = [5, -4, -2] + s[1,2,3]$

$\ell_2: [x, y, z] = [2,0,1] + t[2, -1, -1]$

$[1,2,3] \neq k[2, -1, -1]$; therefore, these lines are not parallel and either intersect once or not at all (skew).

Write in parametric form:

$$\ell_1: \begin{cases} x = 5 + s \\ y = -4 + 2s \\ z = -2 + 3s \end{cases} \qquad \ell_2: \begin{cases} x = 2 + 2t \\ y = -t \\ z = 1 - t \end{cases}$$

Equate the variables:

① $5 + s = 2 + 2t$ ② $-4 + 2s = -t$ ③ $-2 + 3s = 1 - t$

① $3 = 2t - s$ ② $4 = t + 2s$ ③ $3 = t + 3s$

Solve for t and s using elimination with any pair of two equations:

① $3 = 2t - s$

2 ×② $8 = 2t + 4s$ −

$-5 = -5s$

$s = 1$

Sub $s = 1$ in to ① or ②

$3 = 2t - 1$

$t = 2$

Now you MUST check to see if these values of s and t satisfy equation ③ as well:

LS	RS
$= 3$	$= 2 + 3(1)$
	$= 5$

LS $\neq$ RS. Therefore, these lines are skew and do NOT intersect. No solution.

https://www.geogebra.org/3d/jtjbgq55

Part 3: Distance Between Two Skew Lines

The shortest distance between skew lines is the length of the common perpendicular. It can be calculated using the formula $d = \left|\frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{|\vec{n}|}\right|$, where P_1 and P_2 are any points on each line and $\vec{n} = \vec{m}_1 \times \vec{m}_2$ is a normal common to both lines.

Notice this comes from the projection formula learned in the previous unit:

$$\left|proj_{\vec{b}}\, \vec{a}\right| = \left|\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right|$$

Example 3: Determine the distance between the skew lines.

$\ell_1: [x, y, z] = [5, -4, -2] + s[1,2,3]$

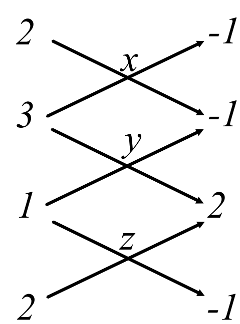
$\ell_2: [x, y, z] = [2,0,1] + t[2, -1, -1]$

Start by calculating $\overrightarrow{P_1P_2}$:

$\overrightarrow{P_1P_2} = [2,0,1] - [5, -4, -2]$

$\overrightarrow{P_1P_2} = [-3,4,3]$

Find a common normal vector by crossing the two direction vectors:



$\vec{n} = [1,2,3] \times [2, -1, -1]$

$\vec{n} = [2(-1) - 3(-1), 3(2) - 1(-1), 1(-1) - 2(2)]$

$\vec{n} = [1,7, -5]$

$$d = \left|\frac{\overrightarrow{P_1P_2} \cdot \vec{n}}{|\vec{n}|}\right|$$

$$d = \left|\frac{[-3,4,3] \cdot [1,7, -5]}{\sqrt{(1)^2 + (7)^2 + (-5)^2}}\right|$$

$$d = \left|\frac{10}{\sqrt{75}}\right|$$

$$d = \frac{10}{5\sqrt{3}}$$

$$d = \frac{2}{\sqrt{3}}$$

$d \cong 1.15$ units

https://www.geogebra.org/3d/sj5mmwuh