L6 – Applications of Rates of Change Unit 1
MCV4U
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Part 1: Rates of Change Applications

Example 1: Suppose the function $V(t) = \frac{50\,000 + 6t}{1+0.4t}$ represents the value, V, in dollars, of a new car t years after it is purchased.

a) What is the rate of change of the value of the car at 2 years? 5 years? And 7 years?

$$V'(t) = \frac{6(1+0.4t) - 0.4(50\,000 + 6t)}{(1+0.4t)^2}$$

$$V'(t) = \frac{6 + 2.4t - 20\,000 - 2.4t}{(1+0.4t)^2}$$

$$V'(t) = \frac{-19994}{(1+0.4t)^2}$$

$$V'(2) = \frac{-19994}{[1+0.4(2)]^2} \cong -6170.99 \text{ \$/year}$$

$$V'(5) = \frac{-19994}{[1+0.4(5)]^2} \cong -2221.56 \text{ \$/year}$$

$$V'(7) = \frac{-19994}{[1+0.4(7)]^2} \cong -1384.63 \text{ \$/year}$$

b) What was the initial value of the car?

$$V(0) = \frac{50\,000 + 6(0)}{1+0.4(0)} = \$50\,000$$

Example 2: Kinetic energy, K, is the energy due to motion. When an object is moving, its kinetic energy is determined by the formula $K(v) = 0.5mv^2$, where K is in joules, m is the mass of the object, in kilograms; and v is the velocity of the object, in meters per second.

Suppose a ball with a mass of 0.35 kg is thrown vertically upward with an initial velocity of 40 m/s. Its velocity function is $v(t) = 40 - 9.8t$, where t is time, in seconds.

a) Express the kinetic energy of the ball as a function of time.

$$K[v(t)] = K(t) = 0.5(0.35)(40 - 9.8t)^2$$

$$K(t) = 0.175(40 - 9.8t)^2$$

b) Determine the rate of change of the kinetic energy of the ball at 3 seconds.

$$K'(t) = 2(0.175)(40 - 9.8t)(-9.8)$$

$$K'(t) = -3.43(40 - 9.8t)$$

$$K'(3) = -3.43[40 - 9.8(3)]$$

$$K'(3) = -36.358$$

At 3 seconds, the rate of change of kinetic energy of the ball is decreasing by 36.358 J/s.

Linear Density:

The linear density of an object refers to the mass of an object per unit length. Suppose the function $f(x)$ gives the mass, in kg, of the first x meters of an object. For the part of the object that lies between x_1 and x_2, the average linear density $= \frac{f(x_2)-f(x_1)}{x_2-x_1}$. The corresponding derivative function $f'(x)$ is the linear density, the rate of change of mass at a particular length x.

Example 3: The mass, in kg, of the first x meters of wire can be modelled by the function $f(x) = \sqrt{3x+1}$.

a) Determine the average linear density of the part of the wire from $x = 5$ to $x = 8$.

$average\ linear\ density = \frac{f(8)-f(5)}{8-5} = \frac{\sqrt{3(8)+1}-\sqrt{3(5)+1}}{3} = \frac{1}{3}$ or about 0.333 kg/m.

b) Determine the linear density at $x = 5$ and $x = 8$. What do these results tell you about the wire.

$$f'(x) = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$$

$$f'(x) = \frac{3}{2\sqrt{3x+1}}$$

$$f'(5) = \frac{3}{2\sqrt{3(5)+1}}$$

$f'(5) = \frac{3}{8}$ or 0.375 kg/m

$$f'(8) = \frac{3}{2\sqrt{3(8)+1}}$$

$f'(5) = \frac{3}{10}$ or 0.3 kg/m

The linear densities are different. Therefore, the material of which the wire is composed is non-homogenous.

Part 2: Business Applications

Terminology:

- The demand functions, or price function, is $p(x)$, where x is the number of units of a product or service that can be sold at a particular price, p.
- The revenue function is $R(x) = x \cdot p(x)$, where x is the number of units of a product or service sold at a price per unit of $p(x)$.
- The cost function, $C(x)$, is the total cost of producing x units of a product or service.
- The profit function, $P(x)$, is the profit from the sale of x units of a product or service. The profit function is the difference between the revenue function and the cost function: $P(x) = R(x) - C(x)$

Economists use the word marginal to indicate the derivative of a business function.

- $C'(x)$ is the marginal cost function and refers to the instantaneous rate of change of total cost with respect to the number of items produced.
- $R'(x)$ is the marginal revenue function and refers to the instantaneous rate of change of total revenue with respect to the number of items sold.
- $P'(x)$ is the marginal profit function and refers to the instantaneous rate of change of total profit with respect to the number of items sold.

Example 4: A company sells 1500 movie DVDs per month at $10 each. Market research has shown that sales will decrease by 125 DVDs per month for each $0.25 increase in price.

a) Determine a demand (or price) function.

Let x represent number of DVDs sold per month
Let p be the price of one DVD
Let n be the number of $0.25 price increases

Equation 1: $x = 1500 - 125n$
Equation 2: $p = 10 + 0.25n$

Re-write price (p) in terms of number of DVDS sold per month (x):

From Equation 1: $n = \frac{1500-x}{125}$

Sub in to Equation 2: $p = 10 + 0.25\left(\frac{1500-x}{125}\right) = 10 + 0.002(1500 - x) = 10 + 3 - 0.002x = 13 - 0.002x$

The demand (price) function is $p(x) = 13 - 0.002x$. This gives the price for one DVD when x DVDs are sold.

b) Determine the marginal revenue when sales are 1000 DVDs per month.

$R(x) = x \cdot p(x)$

$R(x) = x(13 - 0.002x)$

$R(x) = -0.002x^2 + 13x$

$R'(x) = -0.004x + 13$

$R'(1000) = -0.004(1000) + 13$

$R'(1000) = 9$

When sales are 1000 DVDs, the revenue is increasing at a rate of $9 per additional DVD sold.

c) The cost of producing x DVDs is $C(x) = -0.004x^2 + 9.2x + 5000$. Determine the marginal cost when production is 1000 DVDs per month.

$C'(x) = -0.008x + 9.2$

$C'(1000) = -0.008(1000) + 9.2$

$C'(1000) = 1.2$

When producing 1000 DVDs per month, the cost is increasing by $1.20 for each additional DVD produced.

d) Determine the actual cost of producing the 1001st DVD.

$C(1001) - C(1000) = [-0.004(1001)^2 + 9.2(1001) + 5000] - [-0.004(1000)^2 + 9.2(1000) + 5000]$

$C(1001) - C(1000) = 10201.196 - 10200.00$

$C(1001) - C(1000) = 1.196$

The actual cost of producing the 1001st DVD is $1.196. Notice the similarity between the marginal cost of the 1000th DVD and the actual cost of producing the 1001st DVD. For large values of x, the marginal cost when producing x items is approximately equal to the cost of producing one more item, the $(x + 1)$th item.

e) Determine the Profit and Marginal Profit for the monthly sales of 1000 DVDs.

$P(x) = R(x) - C(x)$

$P(x) = -0.002x^2 + 13x - (-0.004x^2 + 9.2x + 5000)$

$P(x) = 0.002x^2 + 3.8x - 5000$

$P(1000) = [0.002(1000)^2 + 3.8(1000) - 5000]$

$P(1000) = 800$

The profit if 1000 DVDs are sold is $800.

$P'(x) = 0.004x + 3.8$

$P'(1000) = 0.004(1000) + 3.8$

$P'(1000) = 7.80$

If 1000 DVDs are sold, the profit is increasing at a rate of $7.80 per additional DVD sold.