L7 – 5.4 Solve Quadratic Trigonometric Equations
MHF4U
Jensen

A quadratic trigonometric equation may have multiple solutions in the interval $0 \le x \le 2\pi$.

You can often **factor** a quadratic trigonometric equation and then solve the resulting two linear trigonometric equations. In cases where the equation cannot be factored, use the **quadratic formula** and then solve the resulting linear trigonometric equations.

You may need to use a Pythagorean identity, compound angle formula, or double angle formula to create a quadratic equation that contains only a single trigonometric function whose arguments all match.

Remember that when solving a linear trigonometric equation, consider all 3 tools that can be useful:

1. Special Triangles
2. Graphs of Trig Functions
3. Calculator

Part 1: Solving Quadratic Trigonometric Equations

Example 1: Solve each of the following equations for $0 \le x \le 2\pi$

a) $(\sin x + 1)\left(\sin x - \frac{1}{2}\right) = 0$

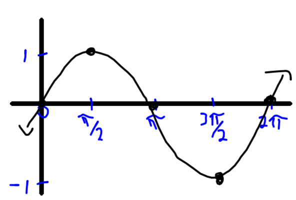
$$(\sin x+1)\left(\sin x - \frac{1}{2}\right) = 0$$

* set both factors equal to zero and solve *

$\sin x + 1 = 0$

$\sin x = -1$ → Graph Yes!

$$\boxed{x_1 = \frac{3\pi}{2}}$$

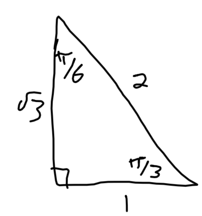


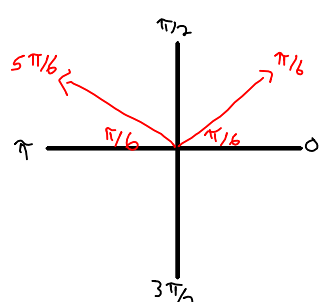
$\sin x - \frac{1}{2} = 0$

$\sin x = \frac{1}{2}$ → Graph No → Δ Yes

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

Put in Q1 + Q2





$$\boxed{x_2 = \pi/6}$$

$x_3 = \pi - \frac{\pi}{6}$

$$\boxed{x_3 = \frac{5\pi}{6}}$$

$$\boxed{\text{Solutions are } x = \frac{3\pi}{2},\ \frac{\pi}{6},\ \text{or } \frac{5\pi}{6} \text{ radians}}$$

b) $\sin^2 x - \sin x = 2$

$\sin^2 x - \sin x = 2$

$\sin^2 x - \sin x - 2 = 0$

Let $\sin x = x$

$x^2 - x - 2 = 0$ P: -2 S: -1 -2 and 1

$(x-2)(x+1) = 0$

$(\sin x - 2)(\sin x + 1) = 0$

$\sin x - 2 = 0$

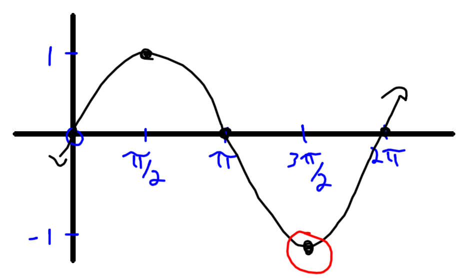
$\sin x = 2$

No solutions

$\sin x + 1 = 0$

$\sin x = -1$ → Graph Yes

$\boxed{x = \frac{3\pi}{2}}$



$\boxed{\text{The only solution is } x = \frac{3\pi}{2}}$

c) $2\sin^2 x - 3\sin x + 1 = 0$

$$2\sin^2 x - 3\sin x + 1 = 0$$

Let $x = \sin x$

$$2x^2 - 3x + 1 = 0 \qquad \begin{matrix} P: 2 \\ S: -3 \end{matrix} \quad -2 \text{ and } -1$$

$$2x^2 - 2x - 1x + 1 = 0$$

$$(2x^2 - 2x) + (-1x + 1) = 0$$

$$2x(x-1) - 1(x-1) = 0$$

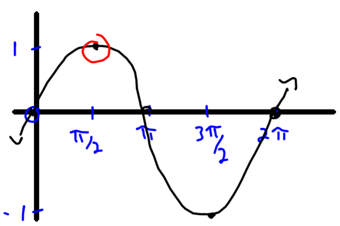
$$(x-1)(2x-1) = 0$$

$$(\sin x - 1)(2\sin x - 1) = 0$$

$$\sin x - 1 = 0$$

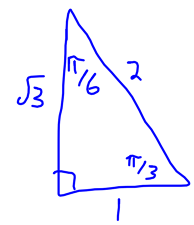
$$\sin x = 1 \leftarrow \text{Graph}$$

$$\boxed{x_1 = \frac{\pi}{2}}$$



$$2\sin x - 1 = 0$$

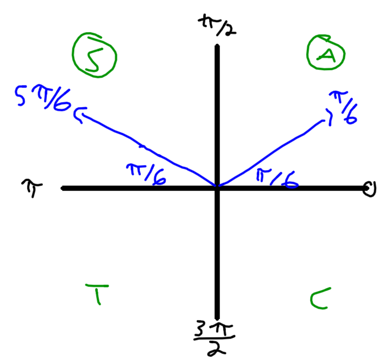
$$\sin x = \tfrac{1}{2} \rightarrow \Delta$$



$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Put in Q1 + Q2

$$x_2 = \frac{\pi}{6}$$



$$x_3 = \pi - \frac{\pi}{6}$$

$$\boxed{x_3 = \frac{5\pi}{6}}$$

$$\boxed{\text{The solutions are } x = \frac{\pi}{2}, \frac{\pi}{6}, \text{ or } \frac{5\pi}{6}}$$

Part 2: Use Identities to Help Solve Quadratic Trigonometric Equations

Example 2: Solve each of the following equations for $0 \leq x \leq 2\pi$

a) $2\sec^2 x - 3 + \tan x = 0$

P.I. $\boxed{\sec^2 x = \tan^2 x + 1}$

$2\sec^2 x - 3 + \tan x = 0$

$2(\tan^2 x + 1) - 3 + \tan x = 0$

$2\tan^2 x + 2 - 3 + \tan x = 0$

$2\tan^2 x + \tan x - 1 = 0$

Let $x = \tan x$

$2x^2 + x - 1 = 0$ P: -2 S: 1 (2 and -1)

$2x^2 + 2x - 1x - 1 = 0$

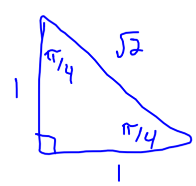
$2x(x+1) - 1(x+1) = 0$

$(x+1)(2x-1) = 0$

$(\tan x + 1)(2\tan x - 1) = 0$

$\tan x + 1 = 0$

$\tan x = -1$ ← Δ



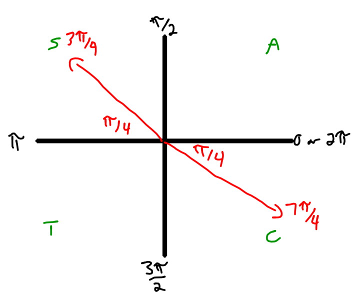
Put $\frac{\pi}{4}$ in Q2 + Q4

$x_1 = \pi - \frac{\pi}{4}$

$\boxed{x_1 = \frac{3\pi}{4}}$

$x_2 = 2\pi - \frac{\pi}{4}$

$\boxed{x_2 = \frac{7\pi}{4}}$



$2\tan x - 1 = 0$

$\tan x = \frac{1}{2}$

$x = \tan^{-1}\left(\frac{1}{2}\right)$

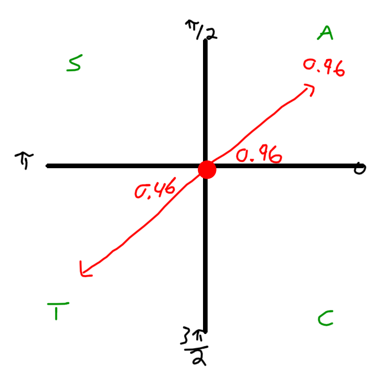
$x \simeq 0.46$ radians

Put in Q1 + Q3

$\boxed{x_3 = 0.46}$

$x_4 = \pi + 0.46$

$\boxed{x_4 = 3.60}$



$\boxed{\text{The solutions are } x = \frac{3\pi}{4}, \frac{7\pi}{4}, 0.46, \text{ OR } 3.60 \text{ RADIANS}}$

b) $3\sin x + 3\cos(2x) = 2$

$$3\sin x + 3\cos(2x) = 2$$
$$3\sin x + 3(1 - 2\sin^2 x) = 2$$
$$3\sin x + 3 - 6\sin^2 x - 2 = 0$$
$$-6\sin^2 x + 3\sin x + 1 = 0$$

Let $x = \sin x$

$$-6x^2 + 3x + 1 = 0$$

P: -6
S: 3

NOT FACTORABLE
Use Q.F.

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(-6)(1)}}{2(-6)}$$

$$x = \frac{-3 \pm \sqrt{33}}{-12}$$

$x \doteq -0.23$

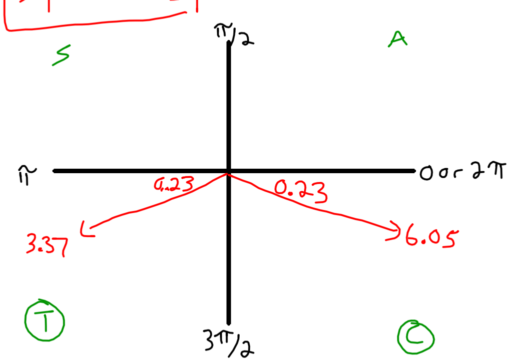
$\sin x = -0.23$ ← CALC

$x_1 = \sin^{-1}(-0.23)$

$x_1 = -0.23$

$x_1 = 2\pi - 0.23$

$\boxed{x_1 = 6.05}$



$x_2 = \pi + 0.23$

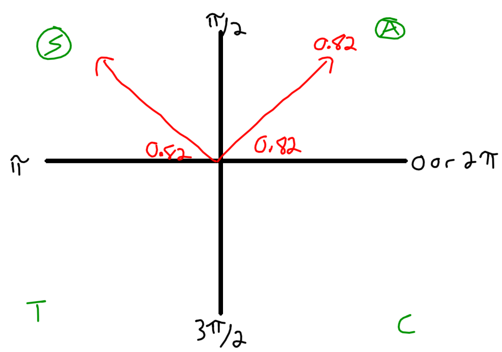
$\boxed{x_2 = 3.37}$

$x \doteq 0.73$

$\sin x = 0.73$ ← CALC

$x = \sin^{-1}(0.73)$

$\boxed{x_1 = 0.82}$



$x_4 = \pi - 0.82$

$\boxed{x_4 = 2.32}$

The solutions are $x = 0.82, 2.32, 3.37,$ or 6.05 RADIANS