L7 – Intersection of Planes Unit 6
MCV4U
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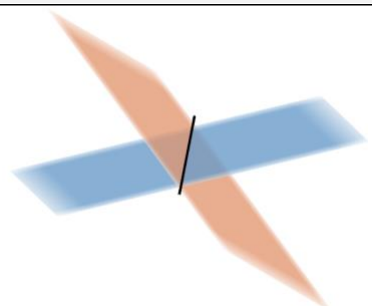
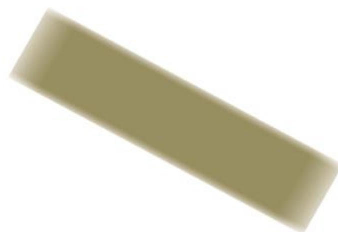
Part 1: Intersection of 2 Planes

Line of Intersection	Infinite Solutions	No Solutions
 If 2 distinct planes intersect, the solution is the set of points that lie on the line of intersection.	 If the planes are coincident, every point on the plane is a solution	 Parallel and distinct planes do not intersect

When solving a system of 2 planes, check if the planes are parallel by analyzing their normal vectors. If they are parallel, determine if they are coincident (infinite solutions) or distinct (no solutions). If the normals are not parallel, that means the planes are not parallel either. You can find the line of intersection by:

i) Eliminating a variable using the method of elimination
ii) Choose 1 of the remaining variables to be the parameter t
iii) Write the other 2 variables in terms of the parameter t as well to get the parametric equation of the line of intersection

Example 1: Describe how the planes in each pair intersect.

a) $\pi_1: 2x - y + z - 1 = 0$

$\pi_2: x + y + z - 6 = 0$

$\vec{n}_1 = [2, -1, 1]$

$\vec{n}_2 = [1, 1, 1]$

$\vec{n}_1 \neq k\vec{n}_2$

The normals are not parallel and therefore the planes are not parallel. This means they intersect in a line.

① $2x - y + z - 1 = 0$

② $x + y + z - 6 = 0$ $\quad +$

$3x + 2z - 7 = 0$

Assign a variable to be the parameter t. Let $z = t$ and then write the other two variables in terms of t.

$3x + 2t - 7 = 0$

$3x = -2t + 7$

$x = -\frac{2}{3}t + \frac{7}{3}$

$x + y + z - 6 = 0$

$\left(-\frac{2}{3}t + \frac{7}{3}\right) + y + t - 6 = 0$

$\frac{1}{3}t + y - \frac{11}{3} = 0$

$y = -\frac{1}{3}t + \frac{11}{3}$

The parametric equations of the line of intersection are:

$$\ell: \begin{cases} x = -\frac{2}{3}t + \frac{7}{3} \\ y = -\frac{1}{3}t + \frac{11}{3} \\ z = t \end{cases}$$

Vector equation is:

$[x, y, z] = \left[\frac{7}{3}, \frac{11}{3}, 0\right] + t[-2, -1, 3]$ Note: the direction vector was simplified

https://www.geogebra.org/3d/jbesfzd5

b) $\pi_3: 2x - 6y + 4z - 7 = 0$

$\pi_4: 3x - 9y + 6z - 2 = 0$

$\vec{n}_3 = [2, -6, 4]$

$\vec{n}_4 = [3, -9, 6]$

$1.5\vec{n}_3 = \vec{n}_4$

The normals are parallel. Therefore, the planes are parallel. They are either coincident or distinct. Solve using elimination to see if you get infinitely many solutions or no solutions.

$3 \times ③ \quad 6x - 18y + 12z = 21$

$2 \times ④ \quad 6x - 18y + 12z = 4 \quad -$

$0 = 17$

This equation is never true. Therefore, there are no solutions and the planes are parallel and distinct.

https://www.geogebra.org/3d/pgsp8f68

c) $\pi_5: x + y - 2z + 2 = 0$

$\pi_6: 2x + 2y - 4z + 4 = 0$

$\vec{n}_5 = [1,1,-2]$

$\vec{n}_6 = [2,2,-4]$

$2\vec{n}_5 = \vec{n}_6$

The normals are parallel. Therefore, the planes are parallel. They are either coincident or distinct. Solve using elimination to see if you get infinitely many solutions or no solutions.

$2 \times ⑤ \quad 2x + 2y - 4z = -4$

$⑥ \quad 2x + 2y - 4z = -4 \quad -$

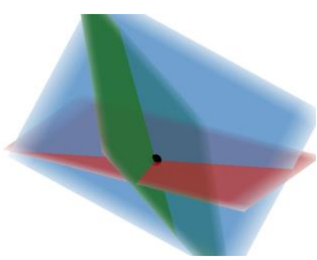
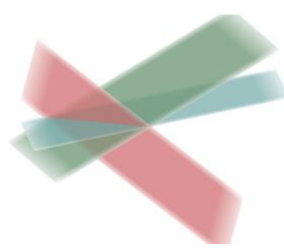
$$0 = 0$$

This equation is always true. Therefore, there are infinitely many solutions. The planes are coincident.

Part 2: Intersection of 3 Planes

A system of three planes is consistent if it has one or more solutions. A system of three planes is inconsistent if it has no solution.

3 scenarios of consistent solutions:

Intersect at 1 Point	**Intersect in a Line**	**Infinite Solutions**
		
Normals are not parallel and not coplanar.	Normals are not parallel but they are coplanar.	Normals are parallel and equations are scalar multiples of eachother.

Normals are parallel if each one is a scalar multiple of the others.

Normals are coplanar if $\vec{n}_1 \cdot \vec{n}_2 \times \vec{n}_3 = 0$

Example 2: For each set of planes, describe the number of solutions and how the planes intersect.

a) π_1: $2x + y + 6z - 7 = 0$
π_2: $3x + 4y + 3z + 8 = 0$
π_3: $x - 2y - 4z - 9 = 0$

Normals are not parallel. Therefore, they intersect at a point or in a line.

Eliminate a variable using two different pairs of equations:

$$\text{①}\quad 2x + y + 6z = 7$$
$$2 \times \text{②}\quad 6x + 8y + 6z = -16 \quad -$$
$$-4x - 7y = 23$$

$$2 \times \text{①}\quad 4x + 2y + 12z = 14$$
$$3 \times \text{③}\quad 3x - 6y - 12z = 27 \quad +$$
$$7x - 4y = 41$$

$$7 \times (\text{①} - \text{②})\quad -28x - 49y = 161$$
$$4 \times (\text{①} + \text{③})\quad 28x - 16y = 164 \quad +$$
$$-65y = 325$$
$$y = -5$$

Now that you have a unique solution for one variable, you can sub back in to previous equations to solve for the remaining variables:

$$-4x - 7(-5) = 23$$
$$-4x = -12$$
$$x = 3$$

$$2x + y + 6z = 7$$
$$2(3) - 5 + 6z = 7$$
$$6z = 6$$
$$z = 1$$

The three planes form a consistent system. The point of intersection is $(3, -5, 1)$.

https://www.geogebra.org/3d/mudhuneu

b) π_4: $x - 5y + 2z - 10 = 0$
π_5: $x + 7y - 2z + 6 = 0$
π_6: $8x + 5y + z - 20 = 0$

Normals are not parallel. Therefore, they intersect at a point or in a line.

Eliminate a variable using two different pairs of equations:

$$
\begin{array}{lr}
④ & x - 5y + 2z = 10 \\
⑤ & x + 7y - 2z = -6 \quad + \\
\hline
 & 2x + 2y = 4
\end{array}
$$

$$
\begin{array}{lr}
④ & x - 5y + 2z = 10 \\
2 \times ⑥ & 16x + 10y + 2z = 40 \quad - \\
\hline
 & -15x - 15y = -30
\end{array}
$$

$$
\begin{array}{lr}
-\frac{1}{15} \times (④ - ⑥) & x + y = 2 \\
\frac{1}{2} \times (④ + ⑤) & x + y = 2 \quad - \\
\hline
 & 0 = 0
\end{array}
$$

This means that there are infinitely many values of x and y that satisfy the equations. Therefore, they intersect in a line. Write the parametric equations of the line. Start by letting a variable equal a parameter:

$$x = t$$

$$x + y = 2$$
$$t + y = 2$$
$$y = 2 - t$$

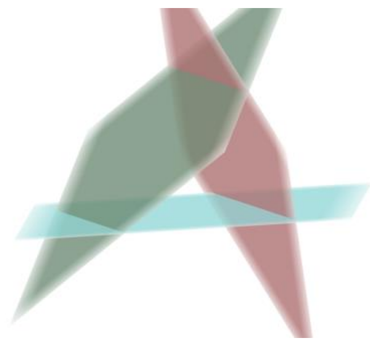
$$x - 5y + 2z = 10$$
$$t - 5(2 - t) + 2z = 10$$
$$t - 10 + 5t + 2z = 10$$
$$2z = -6t + 20$$
$$z = -3t + 10$$

Parametric Equations: $\ell: \begin{cases} x = t \\ y = 2 - t \\ z = 10 - 3t \end{cases}$

Vector Equation: $[x, y, z] = [0,2,10] + t[1, -1, -3]$

https://www.geogebra.org/3d/rvmqunqn

3 scenarios of inconsistent solutions:

2 planes are parallel and the third intersects both of the parallel planes	**The planes intersect in pairs**	**The planes are parallel and at least two are distinct**
		
Two normals are parallel but the third is not	Normals are not parallel but they are coplanar.	Normals are parallel but the equations are not scalar multiples of each other

a) $\pi_1\colon\ 3x + y - 2z = 12$
$\pi_2\colon\ 3x - 5y + z = 8$
$\pi_3\colon\ 12x + 4y - 8z = -4$

π_1 and π_3 are parallel since $4\vec{n}_1 = \vec{n}_2$ but π_2 is not parallel to them. $\pi_1 \neq k\pi_3$, so they are parallel but distinct. This is an inconsistent system with two parallel but distinct planes that are intersected by a third plane.

https://www.geogebra.org/3d/kd3vjrbb

b) $\pi_4\colon\ x + 3y - z = -10$
$\pi_5\colon\ 2x + y + z = 8$
$\pi_6\colon\ x - 2y + 2z = -4$

None of the normals are scalar multiples of each other so none of the planes are parallel. They either intersect at a point, in a line, or not at all. Solve using elimination to determine which scenario it is:

④ $x + 3y - z = -10$
⑤ $2x + y + z = 8$ $\quad +$

$3x + 4y = -2$

$2 \times$ ④ $2x + 6y - 2z = -20$
⑥ $x - 2y + 2z = -4$ $\quad +$

$3x + 4y = -24$

(④ + ⑤) $\quad 3x + 4y = -2$
(④ + ⑥) $\quad 3x + 4y = -24 \quad -$

$0 = -26$

There are no solutions to this equation. Therefore, this is an inconsistent system with no solutions. The planes must intersect in pairs.

https://www.geogebra.org/3d/rnycrtv3

c) π_7: $4x - 2y + 6z = 35$
π_8: $-10x + 5y - 15z = 20$
π_9: $6x - 3y + 9z = -50$

Notice that the normals between each pair of planes are constant multiples of each other. Therefore, the planes are all parallel. However, none of the planes are scalar multiples of each other so they are all parallel but distinct. There are no points of intersection.

https://www.geogebra.org/3d/r3ysp3gm