# Unit 2 Pretest – Curve Sketching

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SOLUTIONS

**1)** Find the increasing and decreasing intervals for each function.

**a)** $f(x) = 7 + 6x - x^2$

$f'(x) = 6 - 2x$

$0 = 6 - 2x$

$x = 3$

$-\infty$ 3 $\infty$

| | | |
|---|---|---|
| | 2 | 4 |
| $f'(x)$ | + | − |
| $f(x)$ | inc. ↗ | dec. ↘ |

increasing: $x < 3$

decreasing: $x > 3$

**b)** $y = x^3 - 48x + 5$

$y' = 3x^2 - 48$

$0 = 3x^2 - 48$

$x^2 = 16$

$x = \pm 4$

$-\infty$ −4 4 $\infty$

| | | | |
|---|---|---|---|
| | −5 | 0 | 5 |
| $y'$ | + | − | + |
| $y$ | inc. ↗ | dec. ↘ | inc. ↗ |

increasing: $x < -4, x > 4$

decreasing: $-4 < x < 4$

**c)** $g(x) = x^4 - 18x^2$

$g'(x) = 4x^3 - 36x$

$0 = 4x(x^2 - 9)$

$0 = 4x(x-3)(x+3)$

$x_1 = 0 \quad x_2 = 3 \quad x_3 = -3$

$-\infty$ −3 0 3 $\infty$

| | | | | |
|---|---|---|---|---|
| | −4 | −1 | 1 | 4 |
| $g'(x)$ | − | + | − | + |
| $g(x)$ | dec. ↘ | inc. ↗ | dec. ↘ | inc. ↗ |

increasing: $-3 < x < 0, x > 3$

decreasing: $x < -3, 0 < x < 3$

**d)** $f(x) = x^3 + 10x - 9$

$f'(x) = 3x^2 + 10$

$0 = 3x^2 + 10$

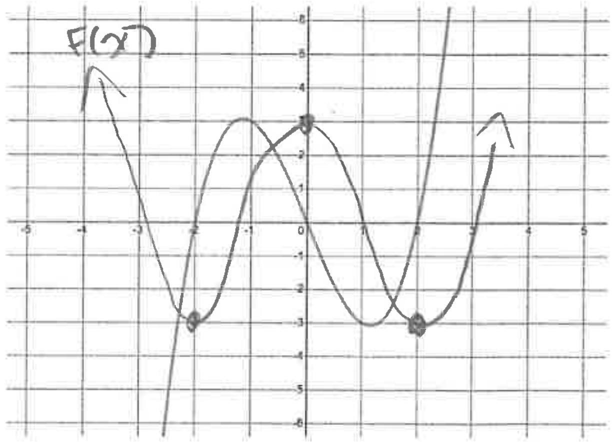
No critical #'s

$f'(x) > 0$ for $x \in \mathbb{R}$, ∴ $f(x)$ is always increasing.

**2)** Given the graph of $f'(x)$, state the intervals of increase and decrease for the function $f(x)$. Then sketch a possible graph of $f(x)$.

increasing: $-2 < x < 0, x > 2$

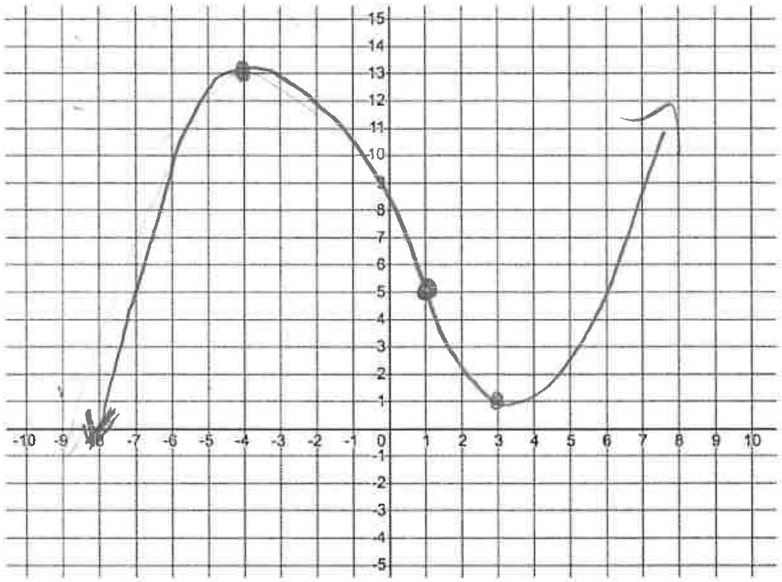
decreasing: $x < -2, 0 < x < 2$



increasing

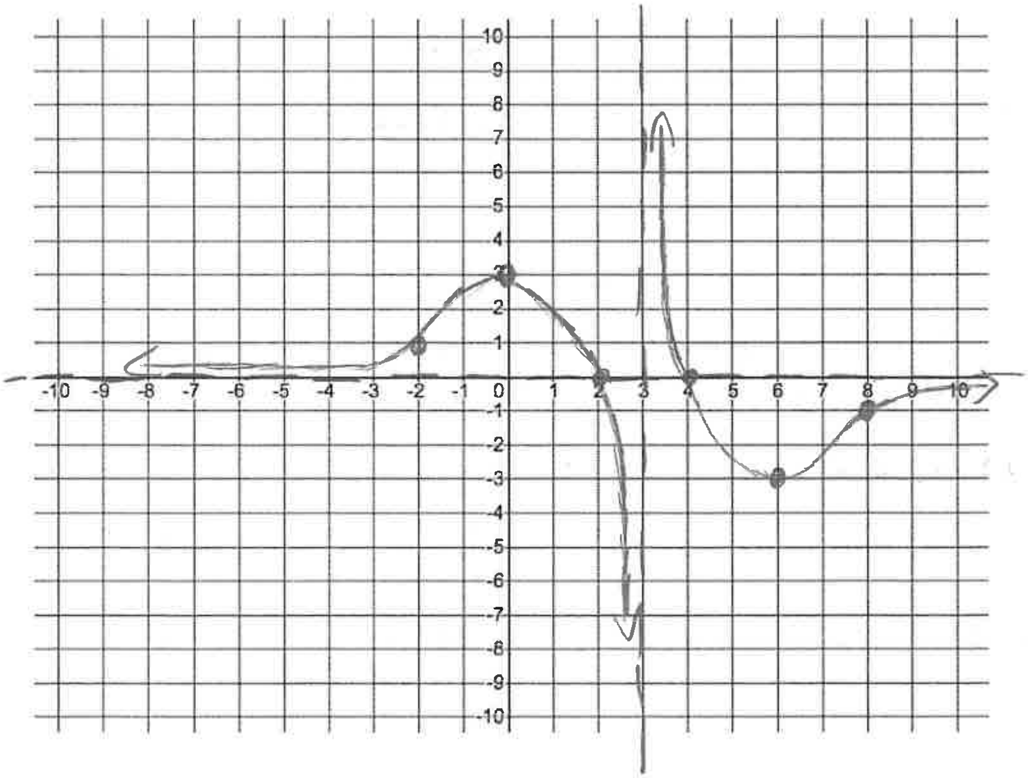
**3)** Sketch a continuous graph that satisfies the following set of conditions: $f'(x) > 0$ when $x < -4$ and $x > 3$, $f'(x) < 0$ when $-4 < x < 3$ and $f(1) = +5$.

decreasing



**4)** Given the following information about $y = f(x)$, sketch a graph for the function on the axes provided below. Label an appropriate scale for the sketch.

Local minimum $(6, -3)$. Local maximum $(0,3)$. Points of inflection at $(-2,1)$ and $(8, -1)$. Increasing when $x < 0$ and $x > 6$; decreasing when $0 < x < 3$ and $3 < x < 6$. Concave up when $x < -2$ and $3 < x < 8$; concave down when $-2 < x < 3$ and $x > 8$. HA at $y = 0$. VA at $x = 3$. $y$-intercept at $(0,3)$. $x$-intercepts at $(2,0)$ and $(4,0)$.



**5)** Find the local extrema for each function and classify them as local max or local min.

**a)** $f(x) = 16 - x^4$

$f'(x) = -4x^3$

$0 = -4x^3$

$x = 0$

$f(0) = 16$

$-\infty \qquad 0 \qquad \infty$

| | -1 | 1 |
|---|---|---|
| $f'(x)$ | + | − |
| $f(x)$ | inc. ↗ | dec. ↘ |

(0,16)
max

Local max at (0,16)

**b)** $g(x) = x^3 + 9x^2 - 21x - 12$

$g'(x) = 3x^2 + 18x - 21$

$0 = x^2 + 6x - 7$

$0 = (x+7)(x-1)$

$x_1 = -7 \qquad x_2 = 1$

$g(-7) = 233 \qquad g(1) = -23$

$-\infty \qquad -7 \qquad 1 \qquad \infty$

| | -8 | 0 | 2 |
|---|---|---|---|
| $g'(x)$ | + | − | + |
| $g(x)$ | inc. ↗ | dec. ↘ | inc. ↗ |

(-7,233) max (1,-23) min

Local max: (-7,233)

Local min: (1,-23)

**6)** The speed, in km/h, of a certain car $t$ seconds after passing a police radar location is given by the function $v(t) = 3t^2 - 24t + 88$.

**a)** Find the min speed of the car.

$v'(t) = 6t - 24$

$0 = 6(t-4)$

$t = 4$

critical point:

$v(4) = 40$

Second derivative test:

$v''(t) = 6$

$v''(4) = 6$ ; ∴ concave up

∴ a local min.

The min speed is 40 km/h

**b)** The radar tracks the car on the interval $2 \leq t \leq 5$. Find the max speed of the car on this interval.

Test endpoints and critical #'s.

$V(2) = 52$ km/h

$v(4) = 40$ km/h

$V(5) = 43$km

Max speed is 52 km/h.

**7)** Determine the absolute extreme values of each function on the given interval.

**a)** $y = x^2 - 3x + 2; -4 \le x \le 4$

$y' = 2x - 3$

$0 = 2x - 3$

$x = \frac{3}{2} = 1.5$

Test endpoints and Critical #'s:

$y(-4) = 30$

$y(1.5) = -0.25$

$y(4) = 6$

absolute max: $(-4, 30)$

absolute min: $(1.5, -0.25)$

**b)** $g(x) = 2x^3 - 24x + 3; -4 \le x \le 2$

$g'(x) = 6x^2 - 24$

$0 = 6(x^2 - 4)$

$0 = 6(x-2)(x+2)$

$x_1 = 2 \quad x_2 = -2$

Tests:

$g(-4) = -29$

$g(-2) = 35$

$g(2) = -29$

absolute max: $(-2, 35)$

absolute min: $(-4, -29)$ and $(2, -29)$

**8)** For the function $f(x) = x^4 - 2x^3 - 12x^2 + 3$, determine the points of inflection and the intervals of concavity.

$f'(x) = 4x^3 - 6x^2 - 24x$

$f''(x) = 12x^2 - 12x - 24$

$0 = 12(x^2 - x - 2)$

$0 = 12(x-2)(x+1)$

$x_1 = 2 \quad x_2 = -1$

Possible PoI:

$f(2) = -45$

$f(-1) = -6$

$-\infty$   $-1$   $2$   $\infty$

| | -2 | 0 | 3 |
|---|---|---|---|
| $f''(x)$ | + | − | + |
| $f(x)$ | C.U.<br>∪ | C.D.<br>∩ | C.U.<br>∪ |

Concave up: $x < -1$, $x > 2$

Concave down: $-1 < x < 2$

PoI $(-1, -6)$

PoI $(2, -45)$

**9)** For the function $f(x) = 2x^3 - x^4$, determine the critical points and classify them using the second derivative test.

$f'(x) = 6x^2 - 4x^3$

$0 = 2x^2(3 - 2x)$

$\downarrow \qquad \downarrow$

$x_1 = 0 \qquad x_2 = \frac{3}{2} = 1.5$

Critical Points:

$f(0) = 0 \quad (0, 0)$

$f(1.5) = \frac{27}{16} = 1.6875 \quad (1.5, 1.6875)$

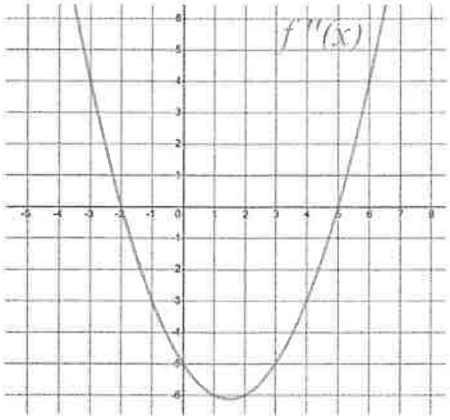
Second Derivative Test:

$f''(x) = 12x - 12x^2$

$f''(0) = 0$ ; test fails, ∴ not a local max or min

$f''(1.5) = -9$; ∴ concave down, local max at $(1.5, 1.6875)$

**10)** Given the graph of $f''(x)$, state the intervals of concavity for $f(x)$.



concave up: $x < -2,\ x > 5$

concave down: $-2 < x < 5$

**11)** For each function, state equations for any asymptotes.

**a)** $f(x) = \frac{x^2-4}{x}$

HA: $y = x$

VA: $x = 0$

**b)** $g(x) = \frac{2x-3}{2x-4}$

HA: $y = 1$

VA: $x = 2$

**c)** $y = \frac{x^2+1}{x^2-3x-10} = \frac{x^2+1}{(x-5)(x+2)}$

HA: $y = 1$
VA: $x = 5, x = -2$

**d)** $\frac{x-1}{x^2+2x+1} = \frac{x-1}{(x+1)^2}$

HA: $y = 0$

VA: $x = -1$

**12)** State the equation of the tangent to the graph of $f(x) = \frac{x+1}{x^2+1}$ at the point where $x = -1$.

Point:

$f(-1) = \frac{(-1)+1}{(-1)^2+1}$

$f(-1) = 0$

$(-1, 0)$

Slope:

$f'(x) = \frac{1(x^2+1) - 2x(x+1)}{(x^2+1)^2}$

$f'(x) = \frac{-1x^2 - 2x + 1}{(x^2+1)^2}$

$f'(-1) = \frac{-1(-1)^2 - 2(-1) + 1}{[(-1)^2+1]^2}$

$f'(-1) = \frac{1}{2}$

$m = \frac{1}{2}$

Eq$^n$:

$y = mx + b$

$0 = (\frac{1}{2})(-1) + b$

$b = \frac{1}{2}$

$$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

**13)** Analyze and sketch each function using the algorithm for curve sketching

**a)** $k(x) = \frac{1}{4}x^4 - \frac{9}{2}x^2$

① No domain restrictions ; no asymptotes

② $0 = \frac{1}{4}x^2(x^2 - 18)$

x-int: (0,0)
(4.24,0)
(-4.24,0)

y-int: (0,0)

$\frac{1}{4}x^2 = 0$  $x^2 - 18 = 0$

$x_1 = 0$  $x_2 = \sqrt{18} \simeq 4.24$

$x_3 = -\sqrt{18} \simeq -4.24$

③ $k'(x) = x^3 - 9x$

$0 = x(x^2 - 9)$

$0 = x(x-3)(x+3)$

$x_1 = 0$  $x_2 = 3$  $x_3 = -3$

$k(0) = 0$  $k(3) = -20.25$  $k(-3) = -20.25$

Critical points: (0,0), (3,-20.25), (-3,-20.25)

④ $k''(x) = 3x^2 - 9$

$0 = 3(x^2 - 3)$

$x = \pm\sqrt{3} \simeq \pm 1.73$

possible PoI: $(\sqrt{3}, -11.25)$ and $(-\sqrt{3}, -11.25)$

(5/6/7)

-∞  -3  -1.73  0  1.73  3  ∞

| | -4 | -2 | -1 | 1 | 2 | 4 |
|---|---|---|---|---|---|---|
| $k'(x)$ | − | + | + | − | − | + |
| $k''(x)$ | + | + | − | − | + | + |
| $k(x)$ | dec-<br>cu | inc<br>cu | inc<br>CD | dec.<br>CD | dec<br>cu | inc<br>cu |

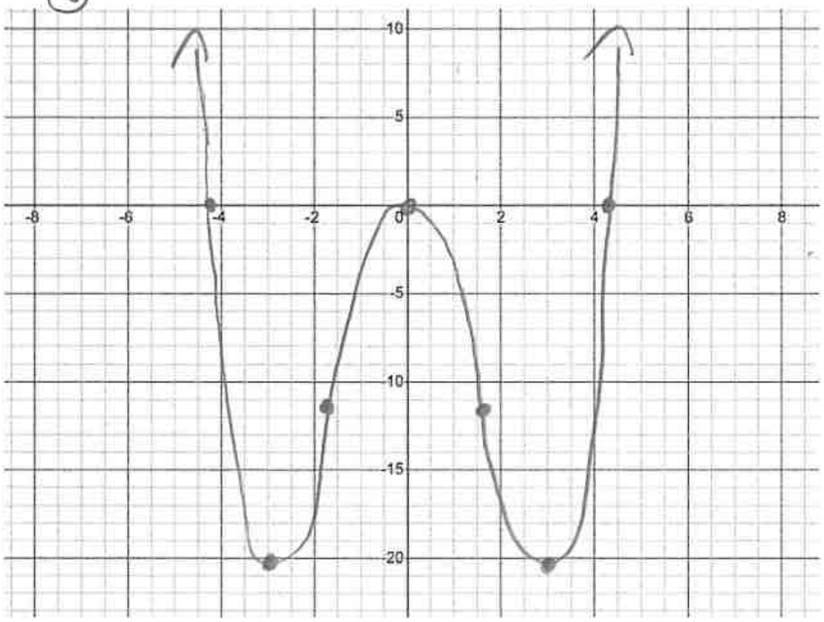
local min  PoI  local max  PoI  local min

Local min: (-3,-20.25) and (3,-20.25)

Local max: (0,0)

PoI: $(-\sqrt{3}, -11.25)$ and $(\sqrt{3}, -11.25)$

⑧



**b)** $h(x) = 2x^3 - 3x^2 - 3x + 2$

① No domain restrictions ; no asymptotes.

② x-int : (-1,0), (2,0), and (0.5,0)  y-int : h(0) = 2

(0,2)

$$\begin{array}{r|rrrr} -1 & 2 & -3 & -3 & 2 \\ & \downarrow & -2 & 5 & -2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

$0 = (x+1)(2x^2 - 5x + 2)$

$0 = (x+1)(2x^2 - 4x - 1x + 2)$

$0 = (x+1)[2x(x-2) - 1(x-2)]$

$0 = (x+1)(x-2)(2x-1)$

$x_1 = -1 \quad x_2 = 2 \quad x_3 = \frac{1}{2}$

③ $h'(x) = 6x^2 - 6x - 3$

$0 = 3(2x^2 - 2x - 1)$

$x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$

$x = \dfrac{2 \pm 2\sqrt{3}}{4}$

$x = \dfrac{2(1 \pm \sqrt{3})}{4}$

$x = \dfrac{1 \pm \sqrt{3}}{2}$

$x_1 \simeq 1.37 \quad x_2 \simeq -0.37$

critical points: (1.37, -2.6), (-0.37, 2.6)

④ $h''(x) = 12x - 6$

$0 = 6(2x - 1)$

$x = \frac{1}{2}$

Possible POI (0.5, 0)

⑤/⑥/⑦

-∞  -0.37  0.5  1.37  ∞

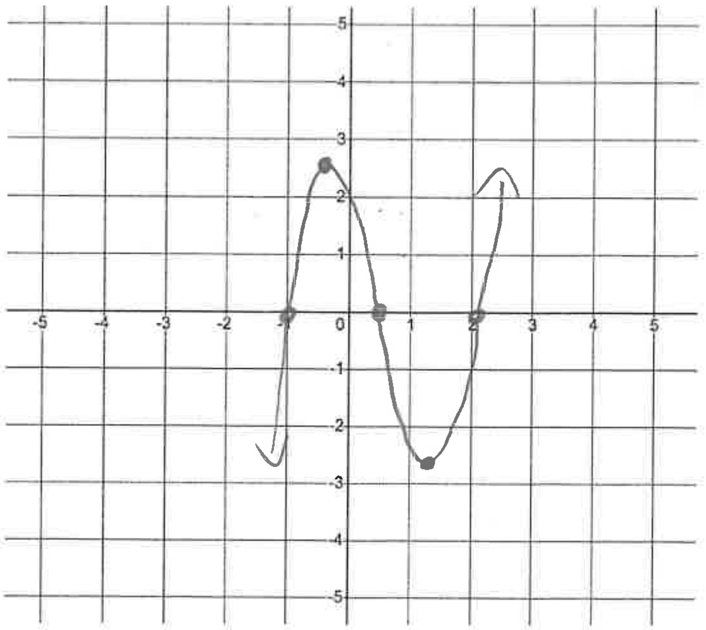
|  | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| h'(x) | + | − | − | + |
| h''(x) | − | − | + | + |
| h(x) | inc.<br>CD | dec.<br>CD | dec.<br>CU. | inc.<br>CU. |

local max  POI  local min

Local min : (1.37, -2.6)

Local max : (-0.37, 2.6)

POI : (0.5, 0)



**c)** $f(x) = \frac{x^2+2x-4}{x^2}$

① HA: $y = 1$

VA: $x = 0$

② x-int: $(1.24, 0)$ and $(-3.24, 0)$

$0 = x^2+2x-4$

$x = \frac{-2 \pm \sqrt{20}}{2}$

$x = \frac{-2 \pm 2\sqrt{5}}{2}$

$x = -1 \pm \sqrt{5}$

$x_1 \simeq 1.24 \quad x_2 \simeq -3.24$

y-int:

$f(0) = \frac{-4}{0}$

∴ no y-intercept.

③ $f'(x) = \frac{(2x+2)(x^2) - 2x(x^2+2x-4)}{x^4}$

$f'(x) = \frac{x[x(2x+2) - 2(x^2+2x-4)]}{x^4}$

$f'(x) = \frac{-2x+8}{x^3}$

$0 = -2x+8$

$x = 4$

critical point: $(4, 1.25)$

④ $f''(x) = \frac{-2(x^3) - 3x^2(-2x+8)}{x^6}$

$f''(x) = \frac{x^2[-2x - 3(-2x+8)]}{x^6}$

$f''(x) = \frac{4x-24}{x^4}$

$0 = 4x-24$

$x = 6$

possible POI: $(6, 1.22)$

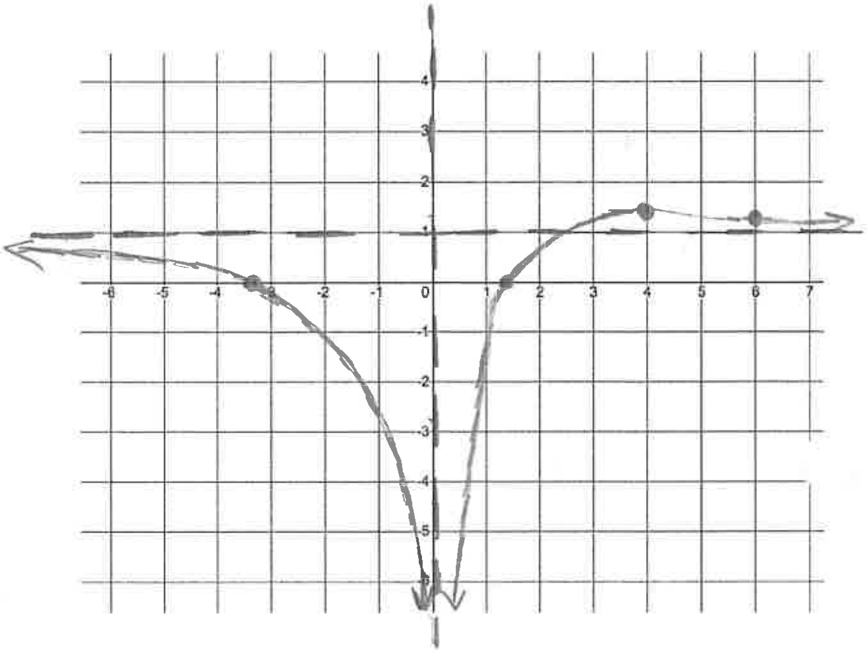
$-\infty \quad 0 \quad 4 \quad 6 \quad \infty$

| | -1 | 1 | 5 | 7 |
|---|---|---|---|---|
| $f'(x)$ | − | + | − | − |
| $f''(x)$ | − | − | − | + |
| $f(x)$ | dec<br>CD | inc.<br>CD | dec<br>CD | dec<br>CU. |

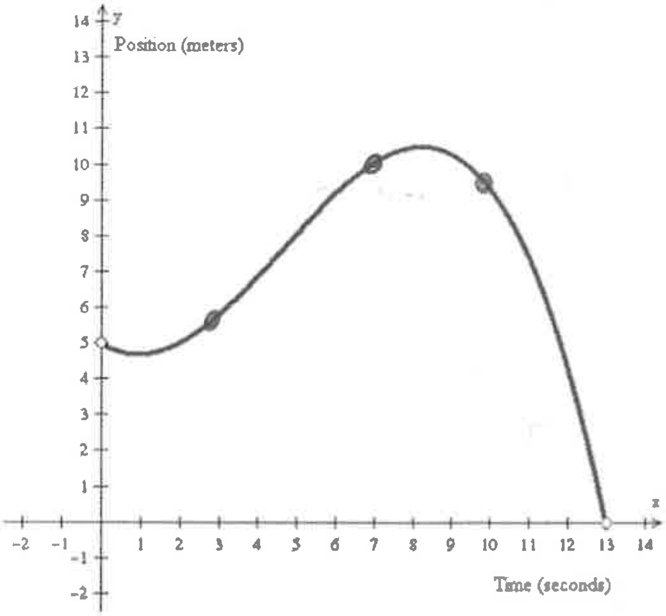
VA

local max<br>
(4, 1.25)

POI<br>
(6, 1.22)



**14)** Consider the graph to the right.



**a)** How many times does the function have a erivative = 0? How do you know?

2 times; 2 turning points.

**b)** State the sign of the first and second derivatives at the following times.

**i)** t = 3 s **ii)** t = 7 s **iii)** t = 10 s

$y'(3) = +$ $y'(7) = +$ $y'(10) = -$

$y''(3) = +$ $y''(7) = -$ $y''(10) = -$

**15)** A garbage can is in the shape of a cylinder with no lid. It needs to have a volume of 5000 cm³. What will be the radius and height of the can that uses the least amount of material to construct it? (Hint: Surface Area)

$V = \pi r^2 h$

$5000 = \pi r^2 h$

$h = \frac{5000}{\pi r^2}$

$SA = \pi r^2 + 2\pi r h$

$SA(r) = \pi r^2 + 2\pi \cancel{r} \left(\frac{5000}{\cancel{\pi} r^{\cancel{2} 1}}\right)$

$SA(r) = \pi r^2 + 10000 r^{-1}$

$SA'(r) = 2\pi r - 10000 r^{-2}$

$SA'(r) = 2\pi r - \frac{10000}{r^2}$

$0 = 2\pi r - \frac{10\,000}{r^2}$

$\frac{10000}{r^2} = 2\pi r$

$\frac{10000}{2\pi} = r^3$

$r = \left(\frac{5000}{\pi}\right)^{1/3}$

$r \simeq 11.7$ cm

$h = \frac{5000}{\pi\left[\left(\frac{5000}{\pi}\right)^{1/3}\right]^2}$

$h \simeq 11.7$ cm

Verify $r \simeq 11.7$ is a max:

$-\infty$ 11.7 $\infty$

| | 10 | 12 |
|---|---|---|
| SA'(r) | − | + |
| SA(r) | dec. ↘ | inc ↗ |

MIN

**MIN** SA when: $r \simeq 11.7$ cm

$h \simeq 11.7$ cm

**16)** A carpenter builds an open box with a square base. She has 8 m$^2$ of wood available. Find the volume of the largest box she can build.

$V = b^2h$

$V(b) = b^2\left(\frac{8-b^2}{4b}\right)$

$V(b) = \frac{b(8-b^2)}{4}$

$V(b) = \frac{8b-b^3}{4}$

$V(b) = 2b - \frac{1}{4}b^3$

$SA = b^2 + 4bh$

$8 = b^2 + 4bh$

$\frac{8-b^2}{4b} = h$

$V'(b) = 2 - \frac{3}{4}b^2$

$0 = 2 - \frac{3}{4}b^2$

$\frac{3}{4}b^2 = 2$

$b^2 = \frac{8}{3}$

$b = \pm\sqrt{\frac{8}{3}}$

$b \simeq 1.63$

2nd derivative test:

$V''(b) = -\frac{3}{2}b$

$V''(1.63) = -2.445$

$\therefore$ $V(1.63)$ is a MAX.

$V(1.63) \simeq 2.18\ m^3$

$\boxed{\text{The max volume is } 2.18\ m^3.}$

**17)** Sandy will make a closed rectangular jewellery box with a square base from two different woods. The wood from the top and botton costs \$0.0020/cm$^2$. The wood for the sides costs \$0.0030/cm$^2$. Find the dimensions that minimize the cost for a box with volume 4000 cm$^3$.

**Constraint:**

$V = x^2h$

$4000 = x^2h$

$h = \frac{4000}{x^2}$

$SA = 2x^2 + 4xh$

$SA(x) = 2x^2 + 4x\left(\frac{4000}{x^2}\right)$

$SA(x) = 2x^2 + 16000x^{-1}$

$C(x) = 0.002(2x^2) + 0.003(16000)x^{-1}$

$C(x) = 0.004x^2 + 48x^{-1}$

$C'(x) = 0.008x - 48x^{-2}$

$0 = 0.008x - \frac{48}{x^2}$

$\frac{48}{x^2} = 0.008x$

$6000 = x^3$

$x \cong 18.17$

**2$^{nd}$ derivative test:**

$C''(x) = 0.008 + 96x^{-3}$

$C''(18.17) \cong 0.024$

Since $C''(18.17) > 0$, $C(18.17)$ is concave UP and $(18.17, C(18.17))$ is a MIN point.

**Answer: The min cost of $C(18.17) = \$3.96$ will occur when the length and width of the box are $18.17$ cm and the height is 12.12 cm.**

**18)** A music store sells an average of 120 CDs per week at \$24 each. A market survey indicates that for each \$0.75 decrease in price, five additional CDs will be sold per week. The cost of producing $x$ CDs is $C(x) = -0.003x^2 + 3x + 2000$. What price and quantity of CDs maximizes profit?

# sold $= x = 120 + 5n \longrightarrow n = \frac{x-120}{5} = 0.2x - 24$

price $= p = 24 - 0.75n$

$p(x) = 24 - 0.75(0.2x - 24)$

$p(x) = 24 - 0.15x + 18$

$p(x) = 42 - 0.15x$

$R(x) = x(42 - 0.15x)$

$R(x) = 42x - 0.15x^2$

$P(x) = (42x - 0.15x^2) - (-0.003x^2 + 3x + 2000)$

$P(x) = -0.147x^2 + 39x - 2000$

$P'(x) = -0.294x + 39$

$0 = -0.294x + 39$

$x \doteq 132.65 \simeq 133$ CD's.

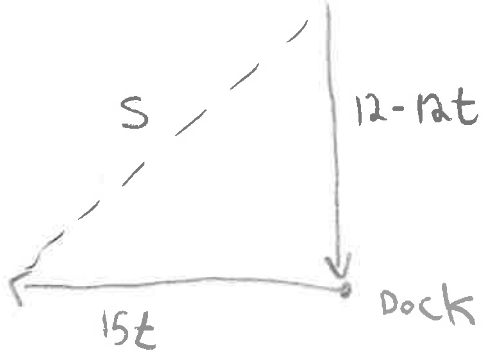
$p(133) = 42 - 0.15(133) = \$22.05$

$P''(133) < 0$; $\therefore$ a max

$\boxed{\text{Selling 133 CD's for \$22.05 each maximizes the profit.}}$

**19)** A boat leaves a dock at 2:00 p.m., heading west at 15 km/h. Another boat heads south at 12 km/h and reaches the same dock at 3:00 p.m. When were the boats closest to each other?

t = hours after 2pm



$$s(t) = \sqrt{(15t)^2 + (12-12t)^2}$$

$$s(t) = \sqrt{225t^2 + 144 - 288t + 144t^2}$$

$$s(t) = (369t^2 - 288t + 144)^{1/2}$$

$$s'(t) = \frac{1}{2}(369t^2 - 288t + 144)^{-1/2}(738t - 288)$$

$$s'(t) = \frac{369t - 144}{(369t^2 - 288t + 144)^{1/2}}$$

$$0 = 369t - 144$$

$$t \approx 0.39 \text{ hours}$$

1st deriv. test

-∞ 0.39 ∞

| | 0.1 | 0.5 |
|---|---|---|
| s'(t) | − | + |
| | dec<br>↘ | inc<br>↗ |

min

∴ The boats were closest together 0.39 hours after 2pm.
(2:23 pm).

## Answers:

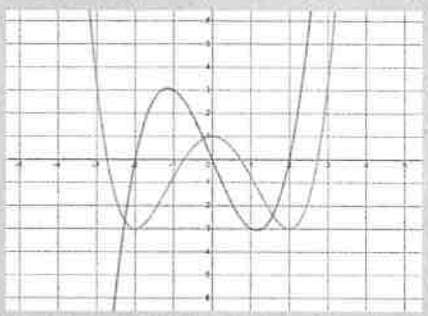
**1)a)** increasing: $(-\infty, 3)$
decreasing: $(3, \infty)$

**b)** increasing: $(-\infty, -4) \cup (4, \infty)$
decreasing: $(-4, 4)$

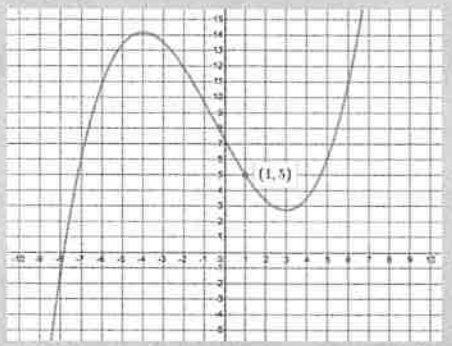
**c)** increasing: $(-3, 0) \cup (3, \infty)$
decreasing: $(-\infty, -3) \cup (0, 3)$

**d)** increasing: $(-\infty, \infty)$
decreasing: never

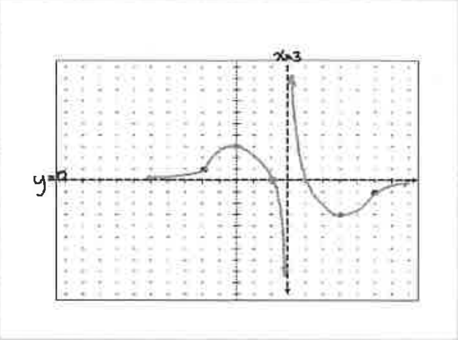
**2)** increasing: $(-2, 0) \cup (2, \infty)$
decreasing: $(-\infty, -2) \cup (0, 2)$



**3)**



**4)**



**5)a)** (0,16) is a local max **b)** local max at (−2,233); local min at (1, −23)

**6)a)** 40 km/h **b)** 52 km/h

**7)a)** absolute min: (1.5, −0.25)
absolute max: (−4, 30)

**b)** absolute min: (−4, −29) and (2, −29)
absolute max: (−2, 35)

**8)** concave up: $x < -1$ and $x > 2$
concave down: $-1 < x < 2$
points of inflection: (−1, −6) and (2, −45)

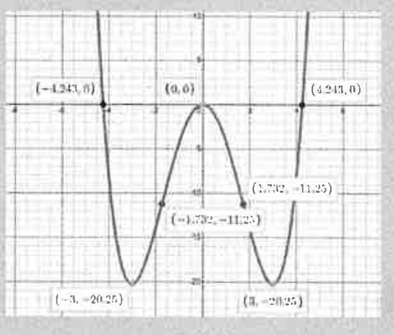
**9)** (0,0) is a point of inflection NOT a local min or max; (1.5,1.7) is a local max

**10)** concave up: $x < -2$, $x > 5$
concave down: $-2 < x < 5$

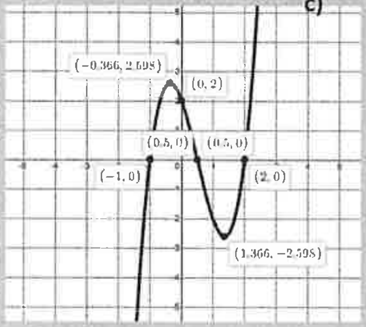
**11)a)** VA: $x = 0$; SA: $y = x$ **b)** VA: $x = 2$; HA: $y = 1$ **c)** VA: $x = 5$ and $x = -2$; HA: $y = 1$
**d)** VA: $x = -1$; HA: $y = 0$

**12)** $y = \frac{1}{2}x + \frac{1}{2}$

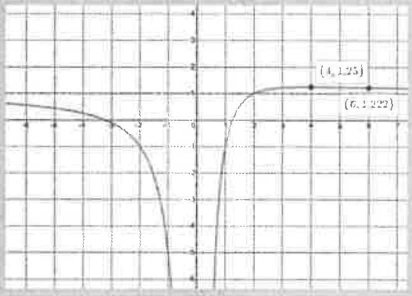
**13)a)**



**b)**



**c)**



**14) a)** twice – there are two turning points for the graph, so two points with $y' = 0$

**b) i)** $y' > 0,\ y'' > 0$ **ii)** $y' > 0,\ y'' < 0$ **iii)** $y' < 0,\ y'' < 0$

**15)** $r = h = 11.7$ cm

**16)** $V \cong 2.17\ \text{m}^3$

**17)** 12.1 cm X 18.2 cm X 18.2 cm

**18)** approximately 133 CDs at a price of $22.05

**19)** 2:23 pm