W3 – 1.6 Instantaneous Rates of Change
MHF4U
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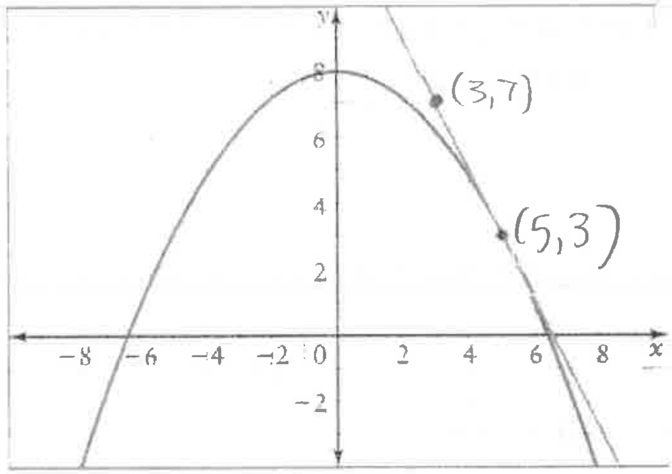
Solutions

1) Consider the graph shown.



a) State the coordinates of the tangent point

(5,3)

b) State the coordinates of another point on the tangent line

(3,7)

c) Use the points you found to find the slope of the tangent line

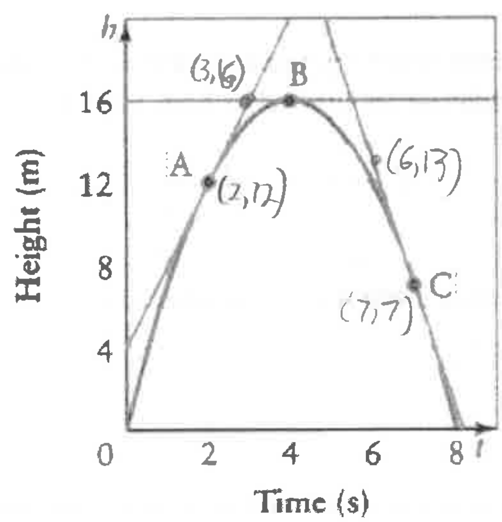
$m = \frac{3-7}{5-3} = \frac{-4}{2} = -2$

d) What does the slope of the tangent line represent?

instantaneous rate of change at $x=5$

2)a) At each of the indicated points on the graph, is the instantaneous rate of change positive, negative, or zero?

A: positive
B: zero
C: Negative



b) Estimate the instantaneous rate of change at points A and C.

A

$\frac{dH}{dt}\Big|_{t=2} \approx \frac{16-12}{3-2} = 4\,m/s$

C

$\frac{dH}{dt}\Big|_{t=7} \approx \frac{13-7}{6-7} = -6\,m/s$

c) Interpret the values in part b) for the situation represented by the graph.

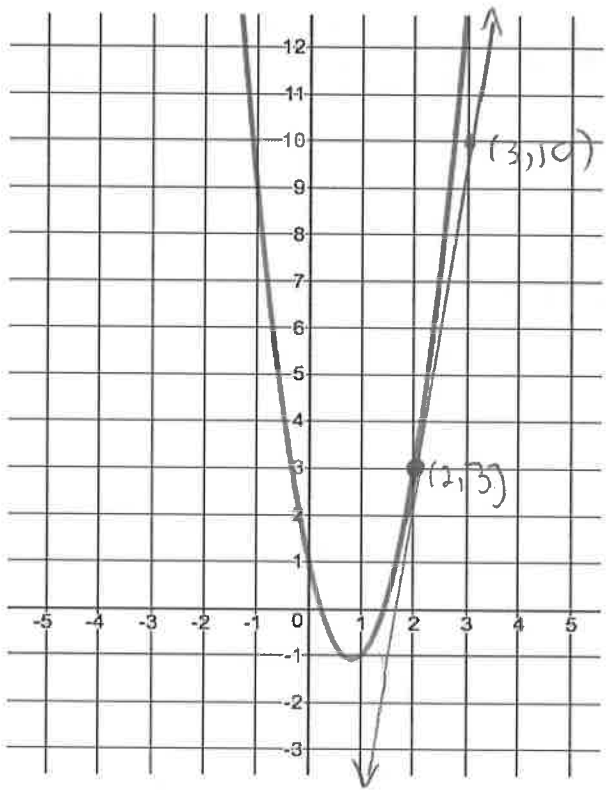
change in distance with respect to time gives a velocity.

3) Use the graph of each function to estimate the instantaneous rate of change at $x = 2$ by drawing a tangent line and calculating it's slope.

$3x^2 - 5x + 1$

$$m = \frac{\Delta y}{\Delta x} = \frac{10-3}{3-2} = 7$$

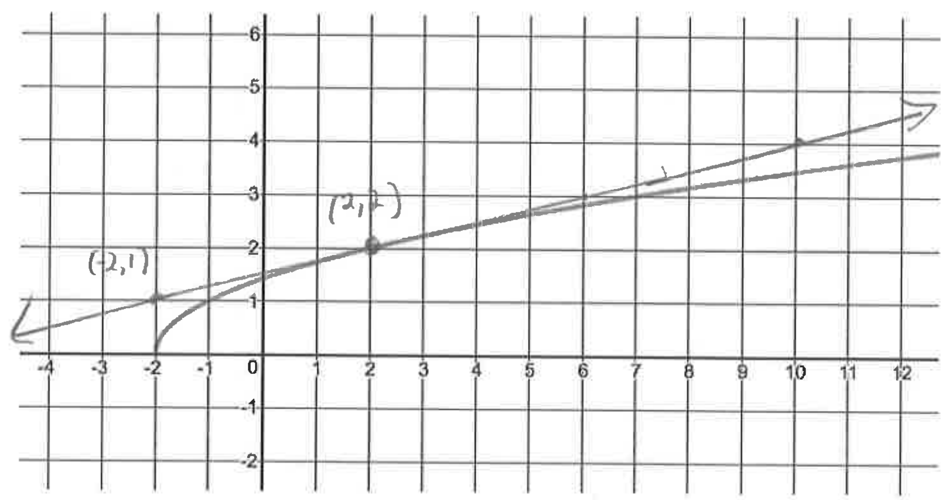
$$\therefore \left.\frac{dy}{dx}\right|_{x=2} \approx 7$$



b) $\sqrt{x+2}$

$$m = \frac{2-1}{2-(-2)} = \frac{1}{4}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=2} \approx \frac{1}{4}$$



4) Verify your answers from question #3 by calculating the LIMIT of the secant slopes as you approach $x = 2$.

a) $3x^2 - 5x + 1$

Interval	Δy	Δx	Slope of secant $= \frac{\Delta y}{\Delta x}$
$2 \le x \le 2.5$	$= f(2.5) - f(2)$ $= 7.25 - 3$ $= 4.25$	$= 2.5 - 2$ $= 0.5$	$= \frac{4.25}{0.5}$ $= 8.5$
$2 \le x \le 2.1$	$= f(2.1) - f(2)$ $= 3.73 - 3$ $= 0.73$	$= 2.1 - 2$ $= 0.1$	$= \frac{0.73}{0.1}$ $= 7.3$
$2 \le x \le 2.01$	$= f(2.01) - f(2)$ $= 3.0703 - 3$ $= 0.0703$	$= 2.01 - 2$ $= 0.01$	$= \frac{0.0703}{0.01}$ $= 7.03$
$2 \le x \le 2.001$	$= f(2.001) - f(2)$ $= 3.007003 - 3$ $= 0.007003$	$= 2.001 - 2$ $= 0.001$	$= \frac{0.007003}{0.001}$ $= 7.003$

$$\left.\frac{dy}{dx}\right|_{x=2} \simeq 7$$

b) $\sqrt{x+2}$

Interval	Δy	Δx	Slope of secant $= \frac{\Delta y}{\Delta x}$
$2 \le x \le 2.5$	$= f(2.5) - f(2)$ $= 2.121320344 - 2$ $= 0.121320344$	$= 2.5 - 2$ $= 0.5$	$= \frac{0.121320344}{0.5}$ $= 0.242640687$
$2 \le x \le 2.1$	$= f(2.1) - f(2)$ $= 2.024845673 - 2$ $= 0.024845673$	$= 2.1 - 2$ $= 0.1$	$= \frac{0.024845673}{0.1}$ $= 0.24845673$
$2 \le x \le 2.01$	$= f(2.01) - f(2)$ $= 2.002498439 - 2$ $= 0.002498439$	$= 2.01 - 2$ $= 0.01$	$= \frac{0.002498439}{0.01}$ $= 0.2498439$
$2 \le x \le 2.001$	$= f(2.001) - f(2)$ $= 2.000249984 - 2$ $= 0.000249984$	$= 2.001 - 2$ $= 0.001$	$= \frac{0.000249984}{0.001}$ $= 0.249984$

$$\left.\frac{dy}{dx}\right|_{x=2} \simeq 0.25$$

5) Use the chart below to estimate the slope of the tangent to the curve $y = \sqrt{2-x}$ at $x = 1$.Have 4 (four) decimal place accuracy in the "slope of secant" column. (4 mks)

Interval	Change in y = Δy	Δx	$\frac{\Delta y}{\Delta x}$ = slope of secant
$0 \le x \le 1$	$= f(1) - f(0)$ $= 1 - 1.414213562$ $= -0.414213562$	$= 1 - 0$ $= 1$	$= \frac{-0.414213562}{1} \simeq -0.4142$
$0.5 \le x \le 1$	$= f(1) - f(0.5)$ $= 1 - 1.224744871$ $= -0.224744871$	$= 1 - 0.5$ $= 0.5$	$= \frac{-0.224744871}{0.5} \simeq -0.4495$
$0.9 \le x \le 1$	$= f(1) - f(0.9)$ $= 1 - 1.048808848$ $= -0.048808848$	$= 1 - 0.9$ $= 0.1$	$= \frac{-0.048808848}{0.1} \simeq -0.4881$
$0.99 \le x \le 1$	$= f(1) - f(0.99)$ $= 1 - 1.004987562$ $= -0.004987562$	$= 1 - 0.99$ $= 0.01$	$= \frac{-0.004987562}{0.01} \simeq 0.4988$
$0.999 \le x \le 1$	$= f(1) - f(0.999)$ $= 1 - 1.000499877$ $= -0.000499877$	$= 1 - 0.999$ $= 0.001$	$= \frac{-0.000499877}{0.001} \simeq -0.4999$

Predicted Slope of the Tangent when $x = 1$ … $\left.\frac{dy}{dx}\right|_{x=1} = -0.5$ (follow the trend in the 4[th] column)

6) The data shows the percent of households that play games over the internet.

'ear	1999	2000	2001	2002	2003
% of Households	12.3	18.2	24.4	25.7	27.9

a) Determine the average rate of change, in percent, of households that played games over the internet from 1999 to 2003.

$$m = \frac{\Delta \text{ \% households}}{\Delta \text{ year}} = \frac{27.9 - 12.3}{2003 - 1999} = \frac{15.6}{4} = 3.9\,\%/\text{year}$$

b) Estimate the instantaneous rate of change in percent of households that played games over the internet in the year 2000. Use the method of averaging a preceding and following interval AND the method of choosing a surrounding interval.

Method 1: averaging

for interval [1999, 2000]

$$m = \frac{dy}{dx} = \frac{18.2 - 12.3}{2000 - 1999} = 5.9\,\%/\text{year}$$

for interval [2000, 2001]

$$m = \frac{dy}{dx} = \frac{24.4 - 18.2}{2001 - 2000} = 6.2\,\%/\text{year}$$

$$\left.\frac{dy}{dx}\right|_{x=2000} \simeq \frac{5.9 + 6.2}{2} = 6.05\%/\text{year}$$

Method 2: Surrounding

for interval [1999, 2001]

$$m = \frac{\Delta y}{\Delta x} = \frac{24.4 - 12.3}{2001 - 1999} = 6.05\,\%/\text{year}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=2000} \simeq 6.05\%/\text{year}.$$

7) Consider the data below describing the height of the world's tallest modern human, Robert Wadlow (1918-1940). At his death at 22 years of age, his height was 8 feet, 11.1 inches.

Age in years	4	8	10	13	16	18	19	21	22
Height in cm	160	190	200	220	240	250	260	268	272

a) Find average rate of change in Wadlow's height between the ages of 4 and 22. Show proper units and notation.

$$m = \frac{\Delta y}{\Delta x} = \frac{272-160}{22-4} = \frac{112}{18} \doteq 6.2 \text{ cm/year}$$

b) Estimate the instantaneous rate of change for Robert Wadlow's height when he was 16 years of age using 2 methods.

Method 1: averaging

for interval [16, 18]

$$m = \frac{\Delta y}{\Delta x} = \frac{250-240}{18-16} = \frac{10}{2} = 5 \text{ cm/year}$$

for interval [13, 16]

$$m = \frac{\Delta y}{\Delta x} = \frac{240-220}{16-13} = \frac{20}{3} = 6.67 \text{ cm/year}$$

$$\left.\frac{dy}{dx}\right|_{x=16} \approx \frac{5+6.67}{2} = 5.835 \text{ cm/year}$$

Method 2: surrounding

for interval [13, 18]

$$m = \frac{\Delta y}{\Delta x} = \frac{250-220}{18-13} = \frac{30}{5} = 6 \text{ cm/year}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=16} \approx 6 \text{ cm/year}.$$

Answer Key

1)a) (5, 3) **b)** (3, 7) **c)** $m = -2$ **d)** instantaneous rate of change at $x = 5$
2)a) at A the instantaneous rate of change is positive, at B the instantaneous rate of change is 0, and at C it is negative **b)** A: $m = 4$ m/s C: $m = -6$ m/s c) velocity at 2 seconds and 7 seconds
3&4)a) $\frac{dy}{dx} = 7$ **b)** $\frac{dy}{dx} = 0.25$
5) $\frac{dy}{dx} = -0.5$
6)a) $\frac{\Delta y}{\Delta x} = 3.9$ %/year **b)** averaging: $\frac{dy}{dx} = 6.05$ %/year surrounding: $\frac{dy}{dx} = 6.05$ %/year
7)a) $\frac{\Delta y}{\Delta x} = 6.2$ cm/year **b)** averaging: $\frac{dy}{dx} = 5.83$ cm/year surrounding: $\frac{dy}{dx} = 6$ cm/year