W7 – Connections with Exponential Models Unit 3

MCV4U

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SOLUTIONS

1) A 100-mg sample of thorium-233 (Th-233) is placed into a nuclear reactor. After 10 min, the sample has decayed to 73 mg. Use the equation $N(t) = N_0 e^{-\lambda t}$ to answer the following questions:

a) Determine the disintegration constant λ for Th-233.

$$73 = 100e^{-\lambda(10)}$$
$$0.73 = e^{-10\lambda}$$
$$\ln(0.73) = -10\lambda$$
$$\lambda \simeq 0.031$$

b) Determine the half-life of Th-233

$$50 = 100e^{-0.031t}$$
$$0.5 = e^{-0.031t}$$
$$\ln(0.5) = -0.031t$$
$$t \simeq 22 \text{ minutes}$$

c) Write the equation that gives the amount of Th-233 remaining as a function of time, in terms of half-life.

$$N(t) = 100\left(\frac{1}{2}\right)^{t/22}$$

d) How fast is the sample decaying after 5 min?

$$N'(t) = 100\left(\frac{1}{2}\right)^{t/22}\ln\left(\frac{1}{2}\right)\left(\frac{1}{22}\right)$$
$$N'(5) = 100\left(\frac{1}{2}\right)^{5/22}\ln\left(\frac{1}{2}\right)\left(\frac{1}{22}\right)$$
$$N'(5) \simeq -2.7 \text{ mg/min}$$

2) Radon-222 (Rn-222) is a radioactive element that spontaneously decays into polonium-218 (Po-218) with a half-life of 3.8 days. The atoms of these two substances have approximately the same mass. Suppose that the initial sample of radon has a mass of 100 mg.

The mass of radon, in milligrams, as a function of time is given by the function $M_{Rn}(t) = M_0\left(\frac{1}{2}\right)^{\frac{t}{3.8}}$, where M_0 is the initial mass of radon and M_{Rn} is the mass of radon after time t, in days.

a) How much radon will remain after

i) 1 day?

$M_{Rn}(1) = 100\left(\frac{1}{2}\right)^{1/3.8}$

$= 83.3\text{ mg}$

ii) 1 week?

$M_{Rn}(7) = 100\left(\frac{1}{2}\right)^{7/3.8}$

$= 27.9\text{ mg}$

b) At what rate is the radon decaying at each of these times? $M_{Rn} = 100\left(\frac{1}{2}\right)^{t/3.8}$

$M_{Rn}'(t) = 100\left(\frac{1}{2}\right)^{t/3.8} \ln\left(\frac{1}{2}\right)\left(\frac{1}{3.8}\right)$

i) $M_{Rn}'(1) = 100\left(\frac{1}{2}\right)^{1/3.8} \ln\left(\frac{1}{2}\right)\left(\frac{1}{3.8}\right) \simeq -15.2$ mg/day

ii) $M_{Rn}'(7) = 100\left(\frac{1}{2}\right)^{7/3.8} \ln\left(\frac{1}{2}\right)\left(\frac{1}{3.8}\right) \simeq -5.1$ mg/day

c) How long will it take for a sample of radon to decay to 25% of its initial mass?

$25 = 100\left(\frac{1}{2}\right)^{t/3.8}$

$0.25 = (0.5)^{t/3.8}$

$\log_{0.5}(0.25) = \frac{t}{3.8}$

$t = 3.8\log_{0.5}(0.25)$

$t = 7.6$ days

3) As radon decays, polonium is produced. The mass of polonium, M_{Po}, in milligrams, as a function of time is given by the function $M_{Po}(t) = M_0\left[1 - \left(\frac{1}{2}\right)^{\frac{t}{3.8}}\right]$, where M_0 is the initial mass of radon and t is time, in days.

* Initial mass of 100mg.

a) How much polonium is there

i) initially?

$M_{Po}(0) = 100\left[1 - \left(\frac{1}{2}\right)^{0}\right]$

$= 0$ mg

ii) after 1 day?

$M_{Po}(1) = 100\left[1 - \left(\frac{1}{2}\right)^{1/3.8}\right]$

$\simeq 16.67$ mg

b) Find the first derivative of this function.

$M_{Po}(t) = 100\left[1 - \left(\frac{1}{2}\right)^{t/3.8}\right]$

$= 100 - 100\left(\frac{1}{2}\right)^{t/3.8}$

$M_{Po}'(t) = 0 - 100\left(\frac{1}{2}\right)^{t/3.8}\ln\left(\frac{1}{2}\right)\left(\frac{1}{3.8}\right)$

$= \frac{-100}{3.8}\left(\frac{1}{2}\right)^{t/3.8}\ln\left(\frac{1}{2}\right)$

4)a) Consider a car shock absorber modelled by the equation $h(t) = e^{-0.5t}\sin t$, where $h(t)$ represents the vertical displacement, in meters, as a function of time, t, in seconds. Determine when the maximum vertical velocity, in m/s, occurs and its value, given that $v(t) = h'(t)$.

$v(t) = e^{-0.5t}(-0.5)\sin t + (\cos t)e^{-0.5t}$

$= e^{-0.5t}(-0.5\sin t + \cos t)$

$a(t) = -0.5e^{-0.5t}(-0.5\sin t + \cos t) + (-0.5\cos t - \sin t)(e^{-0.5t})$

$a(t) = e^{-0.5t}\left[-0.5(-0.5\sin t + \cos t) - 0.5\cos t - \sin t\right]$

$a(t) = e^{-0.5t}\left[0.25\sin t - 0.5\cos t - 0.5\cos t - \sin t\right]$

$a(t) = e^{-0.5t}(-0.75\sin t - \cos t)$

$0 = -\frac{3}{4}\sin t - \cos t$

$\frac{3}{4}\sin t = -\cos t$

$\tan t = -\frac{4}{3}$

$t_1 \simeq -0.927, 5.356$

$t_2 \simeq 2.214$

Test critical #'s AND $t=0$

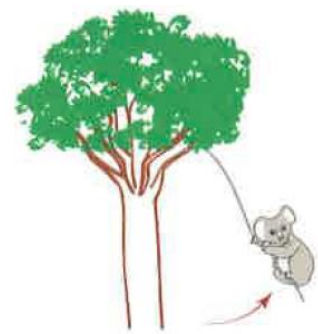
$v(0) = 1$

$v(5.356) \simeq 0.0687$

$v(2.214) \simeq -0.33$

$\boxed{\therefore \text{ Max velocity of } 1\text{ m/s at } t = 0}$

5) Rocco and Biff are two koala bears that are foraging for food together in a eucalyptus tree. Suddenly, a gust of wind causes Rocco to lose his grip and begin to fall. He quickly grabs a nearby vine and begins to swing away from the tree. Rocco's horizontal displacement as a function of time is given by the equation $x(t) = 5\cos\left(\frac{\pi t}{2}\right)e^{-0.1t}$ where x is Rocco's horizontal displacement from the bottom of his swing arc, in meters, at time t, in seconds.



a) Biff can grab Rocco if Rocco swings back to within 1 meter from where he started falling. Will Biff be able to rescue Rocco? Explain, using mathematical reasoning.

Note: period $= \frac{2\pi}{\frac{\pi}{2}} = 4$ seconds

Initial Position: $x(0) = 5\cos(0)e^{0} = 5$ m

Position when he swings back: $x(4) = 5\cos(2\pi)e^{-0.1(4)} \simeq 3.35$ m

He is about 1.65 m away from where he started ; ∴
Biff can NOT rescue Rocco.

b) The other option Rocco has is to let go of the vine at the bottom of one of the swing arcs and drop to the ground. But Rocco will only feel safe doing this if his horizontal velocity at the bottom of the swing is less than 2 m/s. Assuming that Biff is unable to save his friend, how many times must Rocco swing back and forth on the vine before he can safely drop to the ground?

FULL SOLUTION TO THIS QUESTION IS ON THE NEXT PAGE AFTER THE ANSWER KEY

Answers:

1)a) 0.031/min **b)** 22 min **c)** $N(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{22}}$ **d)** -2.65 mg/min

2)a)i) 83.3 mg **ii)** 27.9 mg **b)i)** -15.2 mg/day **ii)** -5.1 mg/day **c)** 7.6 days

3)a)i) 0 mg ii) 16.7 mg **b)** $M'_{Po}(t) = -\frac{100}{3.8}\left(\frac{1}{2}\right)^{\frac{t}{3.8}}\ln\left(\frac{1}{2}\right)$

4)a) $t = 0$ s, $v = 1$ m/s

5)a) He will NOT be able to rescue Rocco. **b)** He must swing back and forth 3.75 times before he can safely drop to the ground.

$$x(t) = 5\cos\left(\frac{\pi t}{2}\right)e^{-0.1t}$$

when is he at the bottom?

$$0 = 5\cos\left(\frac{\pi t}{2}\right)e^{-0.1t}$$

$$0 = \cos\left(\frac{\pi t}{2}\right) \qquad 0 = e^{-0.1t}$$

no solutions

$$\frac{\pi t}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

$$t = 1, 3, 5, 7, \ldots$$

$t = 1 + 2k$ where $k \in \mathbb{Z}$ and $k \geq 0$

\# of times back and forth

$$= \frac{15}{\text{period}}$$

$$= \frac{15}{4}$$

$$= 3.75 \text{ times}$$

velocity at bottom:

$$x'(t) = -5\sin\left(\frac{\pi t}{2}\right)\left(\frac{\pi}{2}\right)e^{-0.1t} + \left(-0.1e^{-0.1t}\right)5\cos\left(\frac{\pi t}{2}\right)$$

$$x'(t) = \frac{-5\pi e^{-0.1t}}{2}\sin\left(\frac{\pi t}{2}\right) - 0.5e^{-0.1t}\cos\left(\frac{\pi t}{2}\right)$$

$$x'(t) = -e^{-0.1t}\left[\frac{5\pi}{2}\sin\left(\frac{\pi t}{2}\right) + 0.5\cos\left(\frac{\pi t}{2}\right)\right]$$

$x'(1) = -7.11$ m/s

$x'(3) = 5.82$ m/s

$x'(5) = -4.76$ m/s

$x'(7) = 3.9$ m/s

$x'(9) = -3.19$ m/s

$x'(11) = 2.61$ m/s

$x'(13) \simeq -2.14$ m/s

$x'(15) = 1.75$ m/s